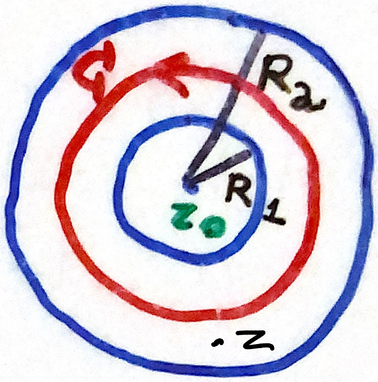
Review: **Laurent Series**

Ex: $f(z) = \frac{1}{z(1-z)} = \frac{1}{z} + 1 + z + z^2 + \cdots + z^n + \cdots$

in the open unit disk (minus 0).

Thm: (Laurent)

Suppose f is analytic throughout an "annular" domain given by



$A = \{ z ; \; \underset{\underset{0}{\vee\!\!|}}{R_1} < |z - z_0| < \underset{\underset{\infty}{\wedge\!\!|}}{R_2} . \}$

Then

1) At each point z in the domain, A $f(z)$ has the series representation

$$f(z) \overset{\circledast}{=} \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} c_n (z-z_0)^n + \sum_{k=1}^{\infty} \frac{c_{-k}}{(z-z_0)^k}$$

2) Let C denote any positively oriented simple closed contour around z_0, lying in the domain, Then

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz .$$

Def: The series $\circledast$ is called a **Laurent series**.

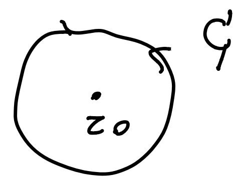
3) The series $\circledast$ is the UNIQE power series centered at z_0, convergent in A, which represents f.

Recall: If $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ is another power series convergent in the annular domain, then

$$\boxed{c_n} \overset{\text{Part (2)}}{=} \frac{1}{2\pi i} \int_C \frac{1}{(z-z_0)^{n+1}} \underbrace{\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k}_{f(z)} \, dz =$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi i} \int_C a_k (z-z_0)^{k-n-1} \, dz$$

$(z-z_0)^{k-n-1}$ has an anti-derivative in $\mathbb{C} \setminus \{z_0\}$ as long as $k-n-1 \neq -1$ ($k \neq n$). So all summands above are zero except the $k=n$ summand.



$$= \frac{1}{2\pi i} \int_C a_n/(z-z_0) \, dz \underset{\text{C.I.F.}}{=} \boxed{a_n}.$$

C.I.F. with the function = constant a_n

Ex: Let $f(z) = \dfrac{4}{z^2-2z-3}$.

We want to find the Laurent Series of f centered at 0 (i.e. powers of z).

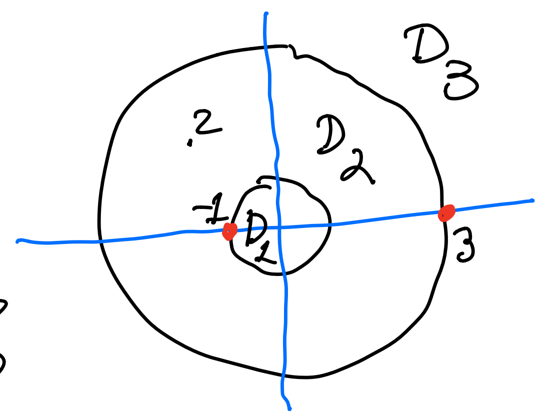
Let us first find the annular domains centered at 0, where f is analytic. $(z^2-2z-3) = (z-3)(z+1)$

f is not analytic at $z=3$, $z=-1$

$D_1 = \{z : |z| < 1\}$

$D_2 = \{z : 1 < |z| < 3\}$

$D_3 = \{z : 3 < |z| < \infty\}$



Partial Fractions:

$$f(z) = \frac{4}{(z-3)(z+1)} = \frac{A}{z-3} + \frac{B}{z+1} =$$

$$= \frac{A(z+1)+B(z-3)}{(z-3)(z+1)}$$

$4 = A(z+1) + B(z-3)$

For $z=3$ get $4 = A(3+1)$, $\boxed{A=1}$

For $z=-1$ get $4 = B(-1-3)$, so $\boxed{B=-1}$.

$$f(z) = \frac{1}{z-3} + \frac{-1}{z+1}.$$

In D_1, f is analytic, so the Laurent Series of f is its Taylor Series (centered at 0).

For $|z/3| < 1$, so for $|z| < 3$.

$$\frac{1}{z-3} = \left(-\frac{1}{3}\right)\left(\frac{1}{1-\frac{z}{3}}\right) = \left(-\frac{1}{3}\right)\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n =$$

$$\frac{1}{z-3} \overset{(*)}{=} \sum_{n=0}^{\infty}\left(-\frac{1}{3^{n+1}}\right)z^n$$

For $|z| < 1$

$$-\frac{1}{z+1} \overset{(**)}{=} -\frac{1}{1-(-z)} = -\sum_{n=0}^{\infty}(-z)^n = \sum_{n=0}^{\infty}(-1)^{n+1}z^n$$

So $$\boxed{f(z) = \sum_{n=0}^{\infty}\left(-\frac{1}{3^{n+1}} + (-1)^{n+1}\right)z^n}$$

In D_2: Eq $(*)$ still holds in D_2.

But $(**)$ is not convergent in D_2

So express $-\frac{1}{z+1}$ as a power series in $\frac{1}{z}$.

$$-\frac{1}{z+1} = -\frac{1}{z}\,\frac{1}{1+\frac{1}{z}} = \left(-\frac{1}{z}\right)\frac{1}{1-\left(-\frac{1}{z}\right)} =$$

$$-\frac{1}{z+1} \overset{***}{=} -\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n \underset{\text{For } |z|>1}{=} -\frac{1}{z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2 - \cdots\right]$$

$$= -\frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 - \cdots$$

$$= \sum_{n=1}^{\infty}(-1)^n \frac{1}{z^n} \underset{k=-n}{=} \boxed{\sum_{k=-\infty}^{-1}(-1)^k z^k}$$

In D_2, $f(z) = \sum_{n=0}^{\infty}\left(-\frac{1}{3^{n+1}}\right)z^n + \sum_{k=-\infty}^{-1}(-1)^k z^k$.

In D_3: Eq $(***)$ holds in D^3.

Eq $(*)$ does not hold, not convergent in D^3.

$$\frac{1}{z-3} \overset{\text{powers of } 1/z}{=} \frac{1}{z}\left(\frac{1}{1-(3/z)}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n$$

Valid for $|3/z|<1$, i.e. for $|z|>3$, so in D_3

$$\frac{1}{z-3} = \sum_{n=0}^{\infty} 3^n z^{-n-1} = \boxed{\sum_{k=-\infty}^{-1} 3^{-k-1} z^k}$$

$k=-n-1$

$n=-k-1$

$$f(z) = \boxed{\sum_{k=-\infty}^{-1} (-1)^k z^k} + \sum_{k=-\infty}^{-1} 3^{-k-1} z^k =$$

$$\overset{\shortparallel}{-\frac{1}{z+1}}$$

$$f(z) = \sum_{k=-\infty}^{-1} \left[(-1)^k + 3^{-k-1} \right] z^k .$$

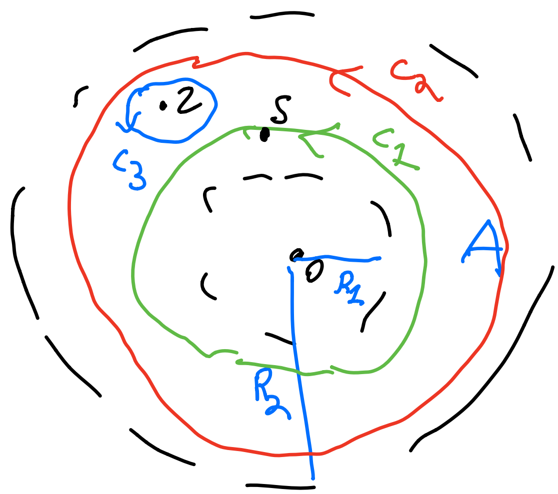
$\uparrow$

in D^3

The idea of the proof of Laurent's Thm; case $z_0=0$

$A = \{z : R_1 < |z| < R_2\}$

$f(z)$ analytic in A.



$$f(z) \overset{C.I.F}{=} \frac{1}{2\pi i}\int_{C_3} \frac{f(s)}{s-z}\,ds. \quad (*)$$

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds = \overbrace{\frac{1}{2\pi i}\int_{C_3} \frac{f(s)}{s-z}\,ds}^{f(z)} + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}\,ds$$

Cauchy-Goursat for Multiply connected domain

$$\text{So,}\quad f(z) = \underbrace{\frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,ds}_{\text{power series in positive powers of } z \text{ as in the proof of Taylor's Thm.}} - \underbrace{\frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s-z}\,ds}_{\text{power series in negative powers of } z}$$

When we express $\boxed{|z| > |s|}$

$$\frac{f(s)}{s-z} = f(s)\frac{1}{s-z}$$ as a power series in z

$$f(s)\left(\frac{1}{z}\right)\frac{1}{1-\left(\frac{s}{z}\right)} = -\frac{f(s)}{z}\sum_{n=0}^{\infty}\left(\frac{s}{z}\right)^{n}$$

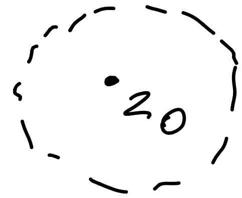
o.k., since $\boxed{\left|\frac{s}{z}\right| < 1}$

Get negative powers of z.

Now integrate term by term.

Ch 6 Residues

<u>Def</u>: A point z_0 is said to be an isolated singular point of a function f, if f is analytic at all points of some disk centered at z_0, except possibly at z_0.



<u>Ex</u>: 0 is an isolated singular point of $1/z$.

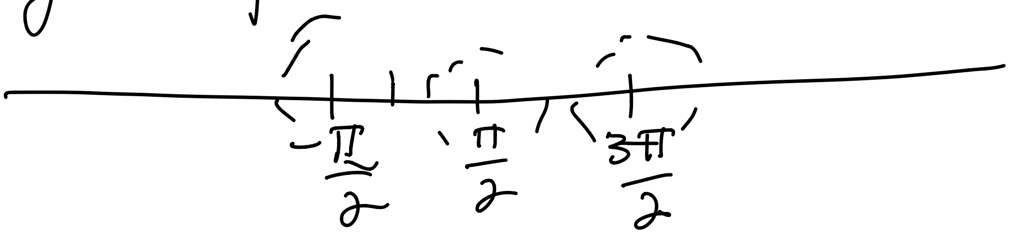
<u>Ex</u>: $f(z) = \dfrac{z-2}{z^4-1}$.

The 4-th roots of 1, i.e. $1, -1, i, -i$ are isolated singular pts of f.

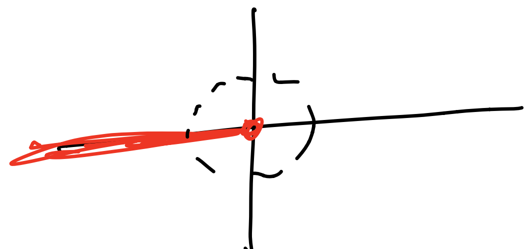
Ex: $f(z) = \tan(z) = \dfrac{\sin(z)}{\cos(z)}$

Zeroes of $\cos(z)$ are at $\frac{\pi}{2} + k\pi$, k integer,

So $f(z)$ has these as isolated singular pts



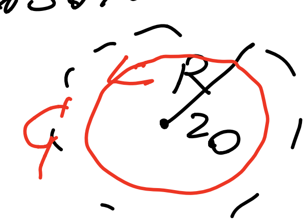
Ex: $f(z) = \mathrm{Log}(z)$



Not analytic on
$\{x : x \text{ is real and} \leq 0\}$,

So 0 is NOT an isolated singular pt.

Note: If z_0 is an isolated singular pt of f,



Then f is analytic in

$D = \{ z : 0 < |z| < R \}$. So Laurent's Theorem apply and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

<u>Def</u>: Let z_0 be an isolated singular point of f. The <u>Residue of f at z_0</u> is the coefficient a_{-1} of $\frac{1}{z - z_0}$ in the Laurent Series of f in the punctured disk centered at z_0.

By Part 2 of Laurent's Thm

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \text{. So}$$

C = circle of radius R sufficiently small centered at z_0.

$$\boxed{a_{-1} = \frac{1}{2\pi i} \int_C f(z) \, dz \text{.}} \quad (\dagger)$$

Ex: Calculate the residue

$$\operatorname{Res}_{\underset{z_0}{\overset{\shortparallel}{z=1}}} \Big(\underbrace{\frac{e^z}{z^2-1}}_{f(z)} \Big) =$$

coef of $\frac{1}{z-1}$ in the power series of f in powers of $z-1$.

$$f(z) = \frac{e^z}{(z-1)(z+1)} = \frac{1}{z-1} \Big(\underbrace{\frac{e^z}{z+1}}_{} \Big)$$

analytic at 1 so has a Taylor series in powers of $(z-1)$

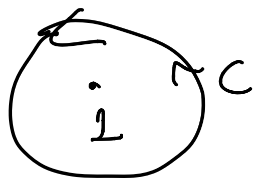
$$\frac{e^1}{2} + b_1 (z-1) + b_2 (z-1)^2 + \vdots$$

$f(z) = \frac{e^1}{2} \cdot \frac{1}{z-1} +$ non neg powers of $(z-1)$

So $\operatorname{Res}_{z=1} f(z) = \frac{e^1}{2}$

Using (f)

$$\operatorname*{Res}_{z=1}\left(\frac{e^z}{(z-1)(z+1)}\right) = \frac{1}{2\pi i}\int_C \frac{\left(\frac{e^z}{z+1}\right)}{z-1}\,dz$$



$$\underset{=}{\text{C.I.F}}\ \frac{e^1}{1+1} = \frac{e}{2}.$$