Integrals of complex valued functions of one real variable:

Def: Let $w(t) = u(t) + i\,v(t)$ be defined on $[a,b]$, with u, v continuous on the interval. Then

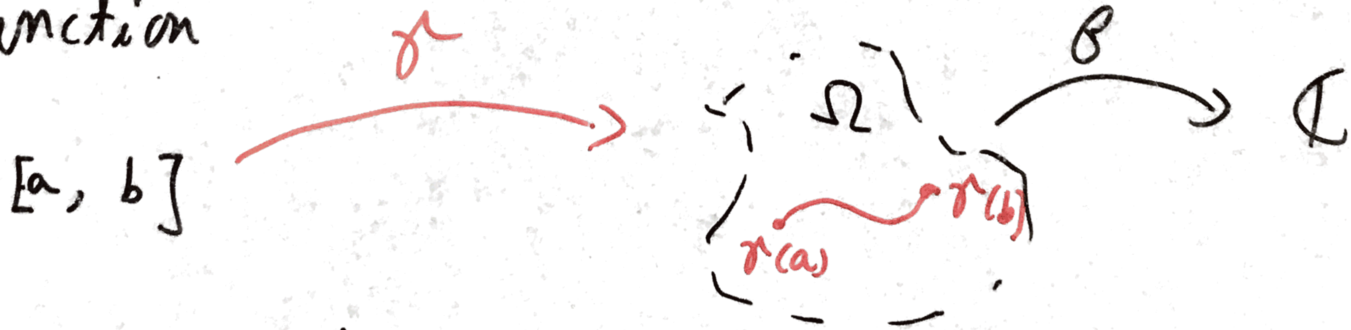
$$\int_a^b w(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Example:

$$\int_0^{\pi} e^{it}\,dt = \int_0^{\pi} \cos(t) + i\sin(t)\,dt =$$

$$\Big[\sin(t)\Big]_0^{\pi} + i\Big[-\cos(t)\Big]_0^{\pi} = 2i$$

Let $\Omega \overset{\text{open}}{\subset} \mathbb{C}$, $f : \Omega \to \mathbb{C}$ an analytic function and $\gamma : [a,b] \to \Omega$ a differentiable function



Lemma 2: $$\int_a^b f'(\gamma(t))\cdot\gamma'(t)\,dt = f(\gamma(b)) - f(\gamma(a))$$

In Leibnitz notation $z(t) = \gamma(t)$

$$\int_a^b \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\,dt = f(z(b)) - f(z(a))$$

Contours:

Def 1: a) A differentiable arc γ (or parametrized curve) is a differentiable map $[a,b] \xrightarrow{\gamma} \mathbb{C}$

$$\gamma(t) = (x(t), y(t))$$

where $x'(t)$, $y'(t)$ exist and are continuous in $[a,b]$.

b) The arc is said to be SMOOTH if $\gamma'(t) \neq 0$ for all t in (a,b).

Def 2: The LENGTH of the smooth arc is

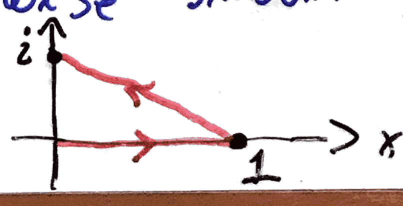
$$L := \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Ex: $\gamma(t) = 5e^{it}$, $0 \le t \le 2\pi$.

$$L = \int_0^{2\pi} |5ie^{it}|\, dt = \int_0^{2\pi} 5\, dt = 10\pi.$$

Def: A CONTOUR, or a piecewise smooth arc, is a finite number of smooth arcs joined end to end.

Example: Find a piecewise smooth parametrization of $\gamma: [0,2] \to \mathbb{C}$ of



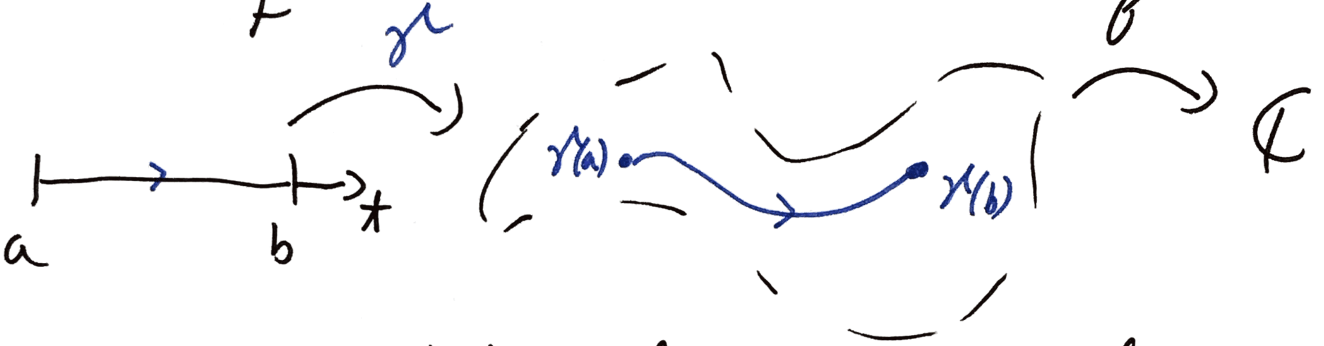
Lemma 1: $\frac{d}{dt}\Big(f(\gamma(t))\Big) = f'(\gamma(t))\cdot\gamma'(t)$

complex multiplication

If $f(z) = F'(z)$

Lemma 2: $\int_a^b \underbrace{f}_{= F'}(\gamma(t))\,\gamma'(t)\,dt = F(\gamma(b)) - F(\gamma(a))$

Def:



The <u>contour integral</u> of a complex valued function f continuous along a contour γ (i.e $f\circ\gamma$ is a continuous func. of t) is:

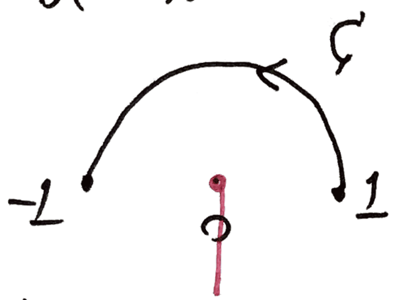
$$\int_\gamma f(z)\,dz \stackrel{\text{def}}{:=} \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt.$$

cpx mult

Sometimes we will not give the parametrization a name $z = z(t)$. We call the contour C,

$$\int_C f(z)\,dz \stackrel{\text{def}}{:=} \int_a^b f(z(t))\cdot \underbrace{z'(t)}_{\frac{dz}{dt}}\,dt$$

Ex: Let C be the contour from 1 to -1 along the upper semi-circle



parametrized by $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$.
Let $f(z) = \frac{1}{z}$. Compute $\int_C f(z)\,dz$.

$$\int_C f(z)\,dz = \int_0^{\pi} \underbrace{\frac{1}{e^{i\theta}}}_{f(z(\theta))} \underbrace{\frac{\partial}{\partial\theta} e^{i\theta}}_{\frac{\partial z}{\partial \theta}} \, d\theta =$$

$$\int_0^{\pi} \frac{1}{\cancel{e^{i\theta}}} \cdot i\, \cancel{e^{i\theta}}\, d\theta = \int_0^{\pi} i \, d\theta = i \int_0^{\pi} 1\, d\theta = \boxed{\pi i}.$$

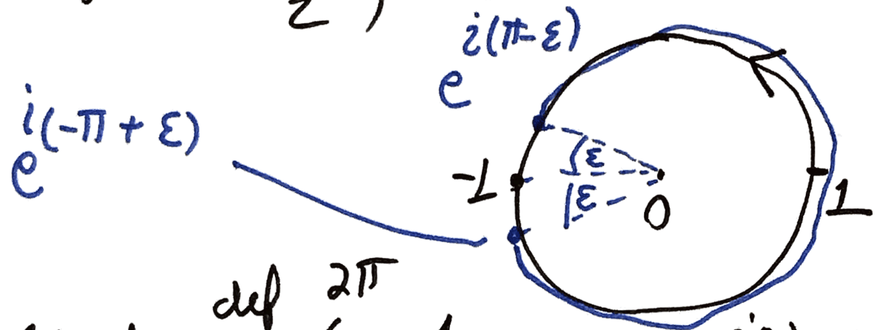
Method 2: (Using Lemma 2) Let $\log(z)$ be the branch of $\log(z)$ with argument in $(-\frac{\pi}{2}, \frac{3\pi}{2})$.
$= \ln|z| + i\arg(z)$

Then $\frac{1}{z} = \frac{\partial}{\partial z}\log(z)$,

$$\int_C \frac{1}{z}\,dz = \int_{\theta=0}^{\theta=\pi} \frac{1}{z(\theta)} z'(\theta)\,d\theta \underset{\text{Lemma 2}}{=} \log(\underbrace{z(\pi)}_{-1}) - \log(\underbrace{z(0)}_{1})$$

$$= \underbrace{\ln|-1|}_{0} + i\underbrace{\arg(-1)}_{\pi} - \underbrace{\log(1)}_{0} = \boxed{\pi i}$$

$\underline{\text{Ex!}}$ $f(z) = \frac{1}{z}$, C = unit circle parametrized by $z(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.



$$\int_C f(z)dz \overset{def}{=} \int_{\theta=0}^{2\pi} \frac{1}{e^{i\theta}} \underbrace{\frac{\partial}{\partial\theta}(e^{i\theta})}_{ie^{i\theta}} d\theta = \int_0^{2\pi} i\, d\theta = \boxed{2\pi i}.$$

Let us compute using Lemma 2: $-\pi \leq \theta \leq \pi$

$$\int_C \frac{1}{z}\, dz \underset{def}{:=} \int_{\theta=-\pi}^{\theta=\pi} \frac{1}{e^{i\theta}} \frac{\partial}{\partial\theta}(e^{i\theta})\, d\theta = \lim_{\varepsilon\to 0} \int_{\theta=-\pi-\varepsilon}^{\theta=\pi-\varepsilon} = \quad d\theta$$

$$\overset{\text{Lemma 2}}{=} \lim_{\varepsilon\to 0} \left[\mathrm{Log}(e^{i(\pi-\varepsilon)}) - \mathrm{Log}(e^{i(-\pi+\varepsilon)}) \right]$$

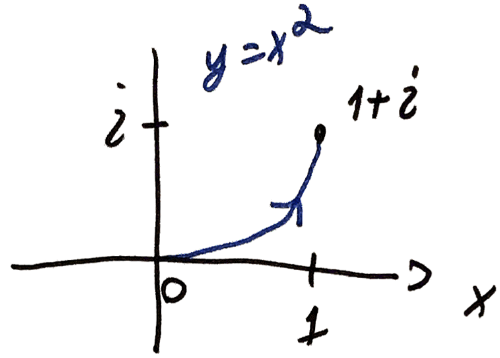
$\uparrow$

$\frac{1}{z} = \frac{\partial}{\partial z}\mathrm{Log}(z)$, and $\mathrm{Log}(z)$ is analytic in a domain $(\mathbb{C} - \mathbb{R}_{\leq 0})$ containing the blue contour,

$$= \lim_{\varepsilon\to 0^+} \underbrace{\left[i\,\mathrm{Arg}(\underbrace{e^{i(\pi-\varepsilon)}}_{\pi-\varepsilon}) - i\,\mathrm{Arg}(\underbrace{e^{i(-\pi+\varepsilon)}}_{-\pi+\varepsilon}) \right]}_{i(2\pi - 2\varepsilon)}$$

$$= \boxed{2\pi i}$$

Ex: $f(z) = z^2$, C is



$$z(t) = t + it^2, \quad 0 \le t \le 1$$

$$\int_C f(z)\,dz \underset{def}{=} \int_{t=0}^{1} (t+it^2)^2 \cdot (\underbrace{1+2ti}_{z'(t)})\,dt =$$

$$[(t^2-t^4)+i(2t^3)]\cdot(1+2ti)$$

$$(\underbrace{t^2-t^4-4t^4}_{t^2-5t^4}) + i\cdot(\underbrace{2t^3-2t^5+2t^3}_{-2t^5+4t^3})$$

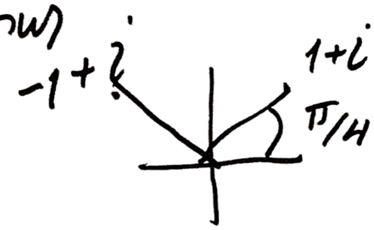
$$= \int_0^1 t^2-5t^4\,dt + i\int_0^1 -2t^5+4t^3\,dt$$

$$\left[\frac{t^3}{3}-t^5\right]_0^1 + i\left[\frac{-t^6}{3}+t^4\right]_0^1 =$$

$$= \left(\frac{1}{3}-1\right) + i\left(-\frac{1}{3}+1\right) = \boxed{-\frac{2}{3}+\frac{2}{3}i}$$

Method using Lemma 2: $\underbrace{z^2}_{f(z)} = \frac{d}{dz}\Big(\underbrace{\frac{z^3}{3}}_{F(z)}\Big)$

$$\int_C z^2\,dz \overset{\text{Lemma 2}}{=} F(\underbrace{1+i}_{\substack{\text{end pt}\\ \text{of contour}}}) - F(\underbrace{0}_{\substack{\text{starting}\\ \text{pt of contour}}}) = \frac{(1+i)^3}{3} - 0 =$$



$$= \frac{1}{3}\left(\sqrt{2}\, e^{\pi i/4}\right)^3 =$$

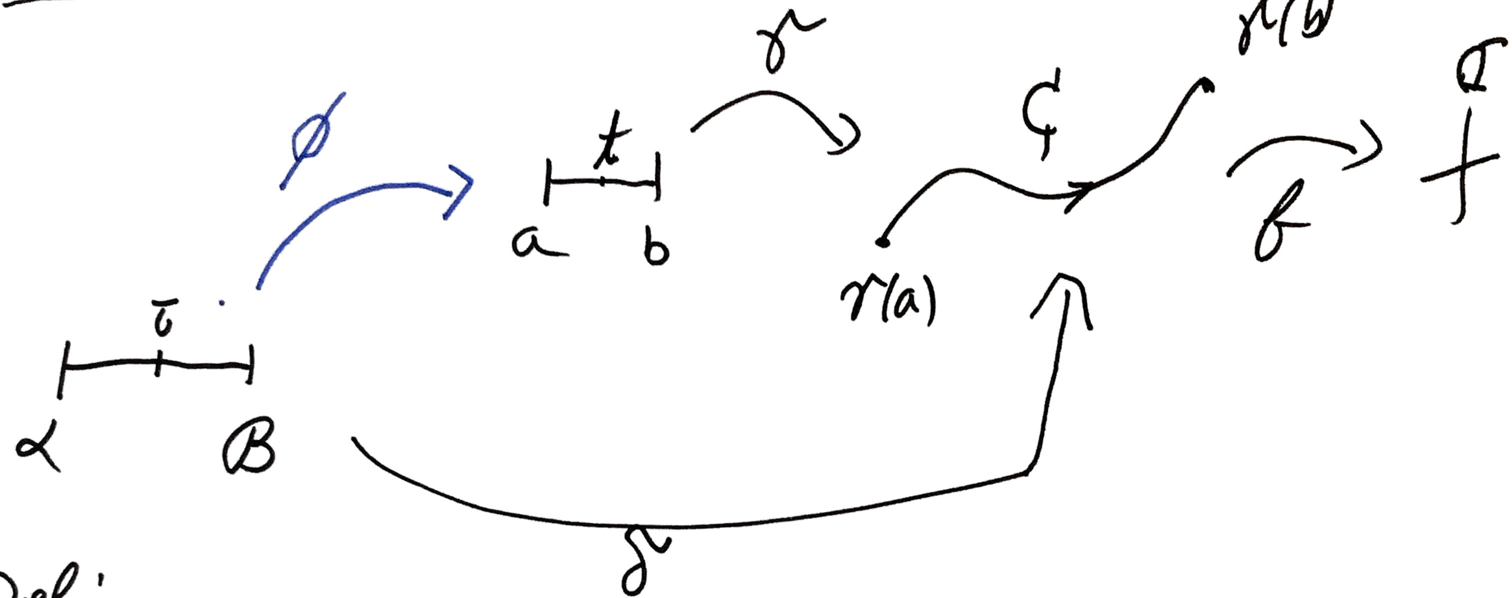
$$= \frac{2\sqrt{2}}{3} e^{3\pi i/4} = \frac{2}{3} \underbrace{\sqrt{2}\, e^{3\pi i/4}}_{-1+i} = \boxed{-\frac{2}{3} + \frac{2}{3} i}$$

Two simple properties of the contour integral:

1) $\int_C (f+g)(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$,

2) $\int_C \underset{\substack{\uparrow\\ \text{complex number}}}{\lambda} f(z)\,dz = \lambda \int_C f(z)\,dz$.

Independence of the contour integral of the parametrization of the path:



Def:

We say that two parametrizations

$$z(t) = \gamma(t), \quad a \le t \le b \quad \text{and}$$

$$z(t) = \delta(\tau), \quad \alpha \le \tau \le \beta$$

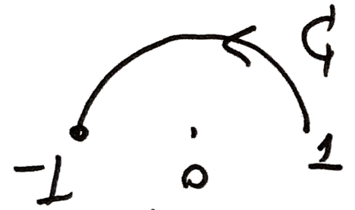
are equivalent parametrization (of a differentiable arc) if there exists a function $\phi : [\alpha, \beta] \to [a, b]$ satisfying

(1) $\delta(\tau) = \gamma(\phi(\tau))$.

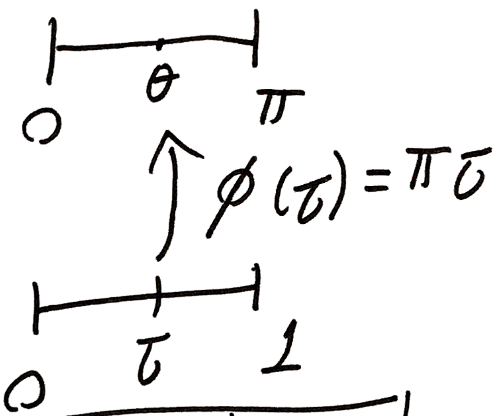
(2) $\phi(\alpha) = a, \quad \phi(\beta) = b$.

(3) $\phi'(\tau) > 0$, for all $\tau \in (a, b)$.

Ex:



upper unit semicircle $\gamma(\theta) = e^{i\theta}$



$$\gamma(\theta) = e^{i\theta}, \quad 0 \le \theta \le \pi.$$

$$\delta(\tau) = e^{i\phi(\tau)} = e^{i\pi\tau}, \quad 0 \le \tau \le 1$$

$$\boxed{\delta(\tau) = \gamma(\phi(\tau)) = e^{i\phi(\tau)} = e^{i\pi\tau}}$$

Lemma: Let f be a complex value func continuous on a contour C_1.

Let $z = \gamma(t)$, $a \le t \le b$ be two equivalent

$z = \delta(\tau)$, $\alpha \le \tau \le \beta$

parametrizations (of C_1). Then

$$\int_{C_1} f(z)\,dz \overset{\text{def}}{=} \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt \overset{\text{def}}{=} \int_\alpha^\beta f(\delta(\tau)) \cdot \delta'(\tau)\,d\tau$$

are equal.