Sec 7.1 page 239 #1:

1(b) $\operatorname{Res}_{z=0} z\cos\left(\frac{1}{z}\right)$

valid on the whole of $\mathbb{C}$

$$\cos(w) = 1 - \frac{w^2}{2} + \frac{w^4}{4!} + \cdots (-1)^k \frac{w^{2k}}{(2k)!}$$

$$\cos\left(\frac{1}{z}\right) = 1 - \frac{1}{2}z^{-2} + \cdots$$

$$z\cos\left(\frac{1}{z}\right) = z - \frac{1}{2}z^{-1} + \cdots$$

$$\operatorname{Res}_{z=0}\left(z\cos\left(\frac{1}{z}\right)\right) = -\frac{1}{2}$$

1(c) $\operatorname{Res}_{z=0}\left(\frac{z-\sin(z)}{z}\right)$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{(-1)^{k-1}}{(2k+1)!} z^{2k+1}$$

valid on the whole of $\mathbb{C}$

$$z - \sin(z) = \frac{z^3}{3!} - \frac{z^5}{5!} \cdots$$

$$\frac{z-\sin(z)}{z} = \frac{z^2}{3!} - \frac{z^4}{5!} + \cdots + \frac{(-1)^{k-1}}{(2k+1)!} z^{2k}$$

We see that the coeff of z^{-1} is 0,

So $\operatorname{Res}_{z=0}\left(\frac{z-\sin(z)}{z}\right) = 0.$

Sec 71 page 239 #2: C = circle of radius 3 centered at 0 oriented counterclockwise.

2(a) $$\oint_C \frac{e^{-z}}{z^2}\,dz = (2\pi i)\underbrace{\operatorname{Res}_{z=0}\left(\frac{e^{-z}}{z^2}\right)}_{\overset{(*)}{=}-1} = -2\pi i$$

$$e^{-z} = 1 - z + \frac{z^2}{2} - \frac{z^3}{3!} + \cdots$$

$$\frac{e^{-z}}{z^2} = \frac{1}{z^2}\boxed{-\frac{1}{z}} + \frac{1}{2} - \frac{z}{3!} + \cdots$$

So, $\operatorname{Res}_{z=0}\left(\frac{e^{-z}}{z^2}\right) \underset{(*)}{=} -1$.

2(c) $$\oint_C z^2 e^{(1/z)}\,dz = 2\pi i \underbrace{\operatorname{Res}_{z=0}\left(z^2 e^{\frac{1}{z}}\right)}_{\overset{(**)}{=}(1/6)} = \frac{\pi i}{3}$$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}z^{-2} + \frac{1}{3!}z^{-3} + \cdots + \frac{1}{m!}e^{-m} + \cdots$$

$$z^2 e^{1/z} = z^2 + z + \frac{1}{2} + \frac{1}{3!}z^{-1} + \cdots$$

So $\operatorname{Res}_{z=0}\left(z^2 e^{1/z}\right) = \frac{1}{6}$

Sec 72 page 243 #1;

a) $z \exp\left(\frac{1}{z}\right)$ is analytic everywhere except at $z=0$.

Its Laurent series centered at 0 in $\mathbb{C} \setminus \{0\}$ is

$$z\left(\sum_{m=0}^{\infty} \frac{z^{-m}}{m!}\right) = \sum_{m=0}^{\infty} \frac{z^{1-m}}{m!} \underset{k=1-m}{=} \sum_{k=-\infty}^{1} \frac{z^{k}}{(1-k)!}$$

The principal part is $\displaystyle\sum_{k=-\infty}^{-1} \frac{z^{k}}{(1-k)!}$. It has infinitely many terms, so $z=0$ is an ESSENTIAL singularity.

b) $\dfrac{z^2}{1+z}$ has an isolated singularity at $z=-1$

The Taylor series of z^2 at $z=-1$ is

$$z^2 = 1 - 2(z+1) + (z+1)^2$$

The Laurent Series of $\dfrac{z^2}{1+z}$ in $\mathbb{C} \setminus \{-1\}$ centered at $z=-1$ is:

$$\frac{1}{z+1} - 2 + (z+1).$$

The principal part has only one term, so finitely many terms, so $z=-1$ is a pole of order 1.

c) $f(z) = \frac{\sin(z)}{z} \underset{\substack{\uparrow \\ \text{valid on } \mathbb{C}\setminus\{0\}}}{=} \frac{1}{z}\left(z - \frac{1}{3!}z^3 + \quad + \frac{(-1)^{k-1} z^{2k+1}}{(2k+1)!} \right)$

$= 1 - \frac{1}{6}z^2 +$ higher degree terms

f is analytic on $\mathbb{C}\setminus\{0\}$ and it's Principal Part at 0 is zero. So $z=0$ is a Removable singularity.

d) $\frac{\cos(z)}{z} \underset{\substack{\uparrow \\ \text{valid on } \mathbb{C}\setminus\{0\}}}{=} \frac{1}{z}\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots + \frac{(-1)^k z^{2k}}{(2k)!} + \cdots\right)$

$= \underbrace{\frac{1}{z}}_{\text{Principal Part.}} - \frac{z}{2} +$ higher degree terms

The " " has one term so we have a pole of order 1 at $z=0$.

e) $f(z) = \frac{1}{(2-z)^3} = -1(z-2)^{-3}$

f is analytic on $\mathbb{C}-\{2\}$, the Principal Part at $z=2$ is the whole of f and consists of one term. So f has a pole of order 3 at $z=2$.

Sec 74 page 248 ~~#~~ #4:

(a) Let C be the circle of radius 2 centered at 0 oriented counterclockwise.

$$I = \int_C \frac{dz}{z^3(z+4)} = 2\pi i \operatorname*{Res}_{z=0} \frac{\left(1/(z+4)\right)}{z^3} = \frac{(2\pi i)\,\phi''(0)}{2!}$$

where $\phi(z) = \frac{1}{z+4} = (z+4)^{-1}$

$$\phi'(z) = -(z+4)^{-2}$$

$$\phi''(z) = 2(z+4)^{-3}$$

$$\phi''(0) = 2(4)^{-3} = 2 \cdot 2^{-6} = \frac{1}{32}$$

So, $I = \frac{\pi i}{32}$.

(b) Let C be the circle of radius 3 centered at -2.

$f(z) = \frac{1}{z^3(z+4)}$ ~~has~~ poles at $z=0$ and $z=-4$, both interior to C, and nowhere else.

Method 1: $I = \int_C \frac{dz}{z^3(z+4)} dz =$

Residue at ∞, Thm page 238

$$2\pi i \operatorname*{Res}_{z=0} \left[\underbrace{\frac{1}{z^2} \frac{1}{\left(\frac{1}{z}\right)^3\left(\frac{1}{z}+4\right)}}_{= \frac{z}{1+4z} \text{ analytic at } z=0} \right] = 0.$$

Method 2:

$$\operatorname*{Res}_{z=-4} \frac{1}{z^3(z+4)} = \frac{1}{(-4)^3} = -\frac{1}{2^6}$$

Residue at a simple pole

Set $f(z) = \frac{1}{z^3(z+4)}$

So $\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-4} f(z) = 0.$

So $I = 0.$