Sec 31 page 97 #10: Show in two ways

that $u(x,y) = \ln(x^2+y^2)$ is Harmonic in every domain

D that does not contain (0,0).

Method 1:

$$u_x = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$u_{xx} = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = 2\frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\left.\begin{aligned} u_y &= \frac{2y}{x^2+y^2} \\ u_{yy} &= 2\frac{x^2-y^2}{(x^2+y^2)^2} \end{aligned}\right\} \text{interchange roles of } x \text{ and } y$$

So the partials are continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ and satisfy $u_{xx}+u_{yy}=0$.

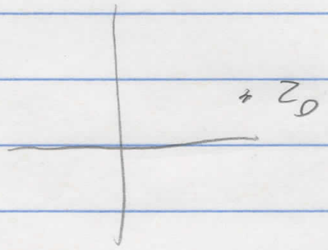
Hence, u is harmonic, by definition, on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Method 2: $\ln(x^2+y^2) = \ln(|x+iy|^2) =$

$\underset{\substack{\uparrow \\ z=x+iy}}{=} \ln(|z|^2) =$ Real part of every branch of $2\log(z)$.

$$\begin{array}{c} 2\log(z) \\ \parallel \\ 2(\ln|z| + i\arg(z)) \end{array}$$

Every point z_0 of $\mathbb{C} \setminus \{(0,0)\}$



is contained in

$\mathbb{C} \setminus$ (non-positive real axis),

where we have the branch of $\log(z)$

with $\arg(z) \in (-\pi, \pi)$,

or in $\mathbb{C} \setminus$ (non-negative part of real axis)

where we have the branch of $\log(z)$

with $\arg(z) \in (0, 2\pi)$. Hence,

every point $z_0 \in \mathbb{C} \setminus \{(0,0)\}$ is contained in a domain D where $\ln(x^2+y^2)$ is the real part of an analytic function, hence harmonic in D, by Theorem 1 of Section 26. Hence, u is Harmonic on $\mathbb{C} \setminus \{(0,0)\}$.