Section 25 page 77 #7:

Let f be analytic everywhere in a domain D. Prove that if f is real valued, then f must be constant throughout D.

Proof: $f(x+iy) = u(x,y) + \underbrace{0}_{V(x,y) \text{ is identically zero}} \cdot i$

Then $f'(x+iy) = \underbrace{u_x(x,y)}_{\boxed{C.R.}\ = v_y(x,y) = 0} + i\,\underbrace{v_x(x,y)}_{=0}$

Hence, $f'(z) = 0$, for every z in D.

The Theorem on page 74 in Sec 24 implies that f is constant in D.

Sec 29 page 92:

$f(z) = e^{(z^2)}$ is entire.

Method 1: The function $h(z) = e^z$ was shown to be entire, as was $g(z) = z^2$. Now $f(z) = h(g(z))$ is the composition of two entire functions, hence it is entire (analytic everywhere), by the Chain Rule. The chain rule asserts also that

$$f'(z) = h'(g(z)) \cdot g'(z)$$

$h'(z) = h(z) = e^z$, $g'(z) = 2z$, so

$$f'(z) = e^{z^2} \cdot (2z)$$

Method 2: $z = x + iy$

$$e^{(z^2)} = e^{(x^2-y^2) + 2xyi} =$$

$$= \underbrace{e^{x^2-y^2} \cos(2xy)}_{u(x,y)} + i \underbrace{e^{x^2-y^2} \sin(2xy)}_{v(x,y)}$$

u and v are defined and have continuous partials everywhere.

$$u_x = e^{x^2-y^2}\cdot(2x)\cos(2xy) + e^{x^2-y^2}(-\sin(2xy))\cdot 2y$$

$$= e^{x^2-y^2}\left(2x\cos(2xy) - 2y\sin(2xy)\right)$$

$$v_y = e^{x^2-y^2}\cdot(-2y)\sin(2xy) + e^{x^2-y^2}\cos(2xy)\cdot 2x$$

$$= e^{x^2-y^2}\left(2x\cos(2xy) - 2y\sin(2xy)\right)$$

So $u_x = v_y$.

Similarly, $u_y = -v_x$.

Hence, $e^{(z^2)}$ is analytic everywhere, by the Theorem in Section 22 page 66.

Section 29 page 92 #6

Show that $|e^{(z^2)}| \overset{*}{\leq} e^{|z^2|}$

Proof: $|e^{(z^2)}| = e^{Re(z^2)} = e^{x^2-y^2}$.

If $w = a+ib$, $a, b \in \mathbb{R}$, then

$$Re(w) = a \leq \sqrt{a^2+b^2} = |w|$$

Taking $w = z^2$ we get

$$Re(z^2) \leq |z^2|, \quad (**)$$

The function $f(x) = e^x$, $f: \mathbb{R} \to \mathbb{R}$ is monotonic increasing (for real x),

so $(**) \Rightarrow$

$$\underbrace{e^{Re(z^2)}}_{\displaystyle = |e^{(z^2)}|} \leq e^{|z^2|}$$

This is inequality $(*)$. $\square$