Sec 38 page 121 problem 2(c):

Let z be a complex number $z = a + ib$, $a > 0$.

$$\int_0^\infty \underbrace{e^{-zt}}\, dt = \lim_{T\to\infty} \int_0^T e^{-zt}\, dt =$$

$$e^{-at}\left[\cos(bt) + i\sin(bt)\right]$$

$$= \lim_{T\to\infty} \left[-\frac{1}{z} e^{-zt}\right]_{t=0}^{t=T} =$$

$$= \left(-\frac{1}{z}\right) \lim_{T\to\infty} \left[e^{-zT} - 1\right]$$

$$= \frac{1}{z} - \frac{1}{z} \lim_{T\to\infty} \left(e^{-zT}\right).$$

It remains to show that the above limit vanishes.

We need to show

$$\lim_{T\to\infty} e^{-aT}\cos(bT) = 0 \text{ and } \lim_{T\to\infty} e^{-aT}\sin(bT) = 0$$

$$-e^{-aT} \leq e^{-aT}\cos(bT) \leq e^{-aT}.$$

$$\underset{T\to\infty}{\downarrow} \quad \downarrow 0 \qquad\qquad \downarrow 0 \quad \underset{T\to\infty}{\downarrow}$$

So $\lim_{T\to\infty} e^{-aT}\cos(bT) = 0$, by the Squeeze Theorem. The proof of the other limit is identical. $\square$

Sec 38 page 121 problem 3:

$$I := \int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \begin{cases} 0 & \text{when } m \neq n \\ 2\pi & \text{when } m = n. \end{cases}$$

Proof:

$$I = \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta.$$

If $m = n$, then $I = \int_0^{2\pi} 1\, d\theta = 2\pi$.

If $m \neq n$, then $e^{i(m-n)\theta} = \frac{\partial}{\partial\theta}\left[\frac{1}{(m-n)i} e^{i(m-n)\theta}\right]$.

So, Eq (4) page 120 yields

$$I = \left[\frac{1}{(m-n)i} e^{i(m-n)\theta}\right]_0^{2\pi}.$$

But $e^{i(m-n)\cdot 2\pi} = e^0$, by the fact that e^z is periodic with period $2\pi i$.

Hence, $I = 0$.

Sec 38 page 121 #4:

$$\int_0^{\pi} \underbrace{e^{(1+i)x}}_{\frac{\partial}{\partial x}\left(\frac{1}{1+i}\right)e^{(1+i)x}} dx = \frac{1}{1+i}\left[e^{(1+i)x}\right]_0^{\pi} =$$

$$= \underbrace{\frac{1}{1+i}}_{\frac{1-i}{2}} \left(\underbrace{e^{(1+i)\pi}}_{-e^{\pi}} - \underbrace{e^{0}}_{1}\right) = \frac{-1-e^{\pi}}{2} + i\left(\frac{e^{\pi}+1}{2}\right)$$

So $\int_0^{\pi} e^{x}\cos(x)\,dx = Re(I) = \frac{-1-e^{\pi}}{2}$

$\int_0^{\pi} e^{x}\sin(x)\,dx = \frac{1+e^{\pi}}{2}$,