Sec 43 page 140 problem 7:

C_R = upper half of circle of radius R, $R>2$, centered at 0 given by
$$z(\theta) = Re^{i\theta}, \quad 0 \le \theta \le \pi.$$

$$f(z) = \frac{2z^2-1}{z^4+5z^2+4} = \frac{2z^2+1}{(z^2+1)(z^2+4)}.$$

Given two complex numbers a, b we have the inequalities

(*) $\quad |a+b| \le |a|+|b|$ and

(**) $\quad |a+b| \ge \big||a|-|b|\big|$.

Hence, if $|z|=R$, then

(1) $\quad |2z^2-1| \le |2z^2| + |-1| = 2R^2+1$, by (*), and

(2) $\quad |(z^2+1)(z^2+4)| \underset{\text{by } **}{\ge} \big||z^2|-|1|\big|\,\big||z^2|-4\big| \underset{R>2}{=} (R^2-1)(R^2-4)$.

Consequently, $|f(z)| \le \underbrace{\dfrac{2R^2+1}{(R^2-1)(R^2-4)}}_{M}$, if $|z|=R>2$, by (1) and (2).

Thus, $$\left|\int_{C} f(z)\,dz\right| \le M \underbrace{\text{length}(C)}_{\pi R} = \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)},$$

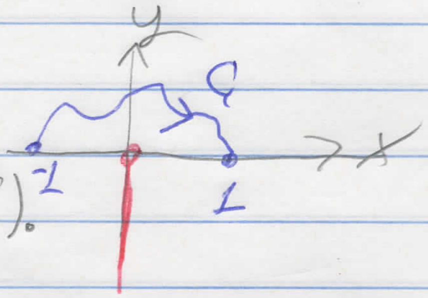
by the Theorem in Sec. 43. ∎

Sec 45 page 149 #3:

Let C_0 be a closed contour not passing through z_0. The function $f(z) = (z-z_0)^{n-1}$ has an antiderivative in $\mathbb{C} \setminus \{z_0\}$, $F(z) = \frac{(z-z_0)^n}{n}$, if $n \neq 0$.

Hence, $\int_{C_0} f(z)\,dz = 0$ by part (c) of the Theorem in Section 44.

Sec 45 page 149 #5:



Let $\log(z)$ be the branch of log with argument in $(-\frac{\pi}{2}, \frac{3\pi}{2})$. It is analytic on the complement Ω of the non-positive part of the y-axis.

If C is a contour from -1 to 1, which lies above the x-axis (except for endpoints), then $\log(z)$ is analytic at all points of C.

Let $f(z) = z^i \overset{def}{:=} e^{i\log(z)}$, for the above branch of log. Then

$$F(z) = \frac{1}{i+1} z^{(i+1)} \underset{def}{:=} \frac{1}{i+1} e^{(i+1)\log(z)}$$

is an antiderivative of f in Ω (using the same branch of log) as shown in class (see also section 33). Hence,

$$\int_C f(z)dz = F(1) - F(-1) =$$

$$= \frac{1}{i+1}\Big(\underbrace{e^{(i+1)\underbrace{\log(1)}_{0}}}_{1} - \underbrace{e^{(i+1)\overbrace{\log(-1)}^{\ln|-1| + i\underbrace{\arg(-1)}_{\pi}}}}_{e^{(i+1)\pi i} = e^{-\pi+\pi i} = -e^{-\pi}}\Big) =$$

$$= \underbrace{\frac{1}{i+1}}_{\frac{1-i}{2}}\left(1 - e^{-\pi}\right).$$

Sec 49 page 160 problem 1 e :

C_1 is the unit circle (with either orientation).

$f(z) = \tan(z) = \dfrac{\sin(z)}{\cos(z)}$. Both sin and cos are entire. Hence, f is analytic away from the zeroes of $\cos(z)$, by the Quotient Rule of differentiation.

cos vanishes precisely at the set $\left\{\frac{\pi}{2} + k\pi \;:\; k \in \mathbb{Z}\right\}$.

Now, $\frac{\pi}{2} > 1$, so $f(z)$ is analytic on $\left\{z : |z| < \frac{\pi}{2}\right\}$ and in particular at all points on and interior to C_1.

Thus, $\displaystyle\int_C f(z)\,dz = 0$, by

Cauchy-Goursat's Theorem.