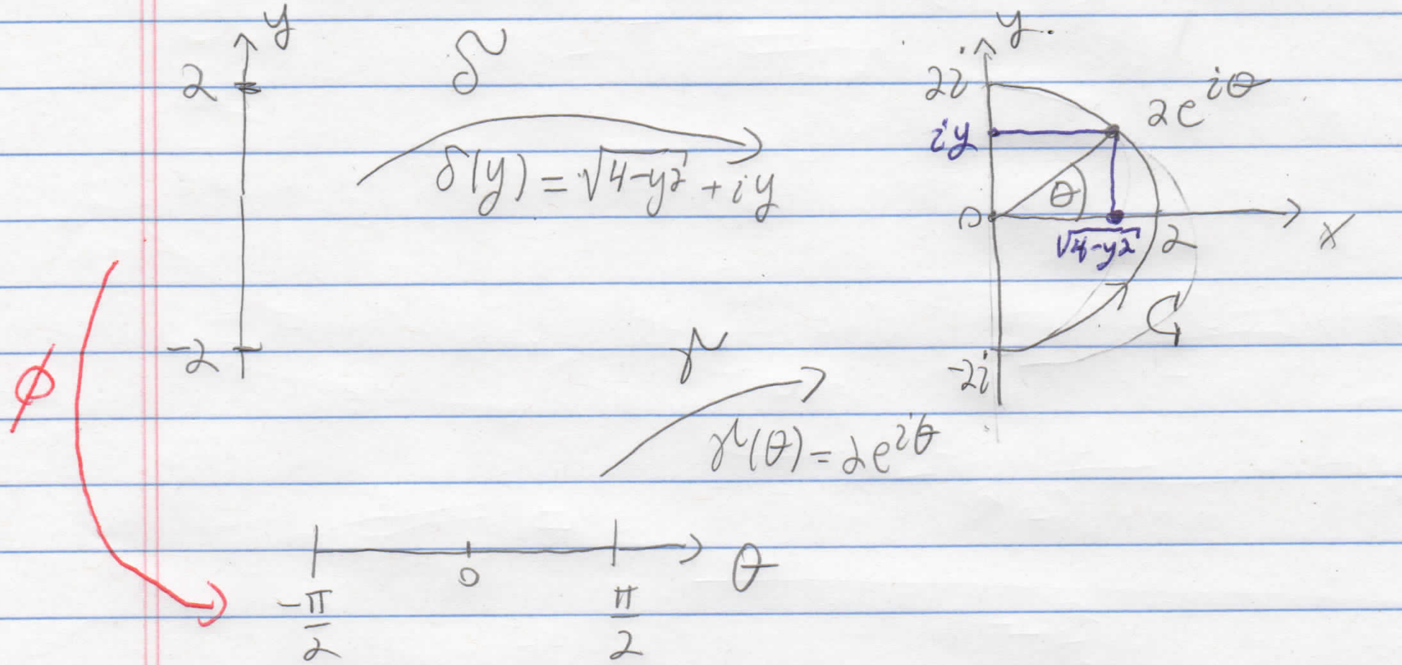
Change notation $\delta(y) = \sqrt{4-y^2} + iy$ (instead of $Z(y)$),
$\gamma(\theta) = 2e^{i\theta}$ instead of $z(\theta)$.

Sec 39 page 125 problem 2:



Define $\phi(y) = \begin{cases} \arctan\left(\frac{y}{\sqrt{4-y^2}}\right), & \text{if } -2 < y < 2, \\ -\pi/2, & \text{if } y = -2, \\ \pi/2 & \text{if } y = 2. \end{cases}$

Check that ϕ is continuous!

Remark: (not used) ϕ is the inverse function of $\psi: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-2, 2]$, given by $\psi(\theta) = 2\sin(\theta)$. It is easy to check that $\gamma(\theta) = \delta(\psi(\theta))$.

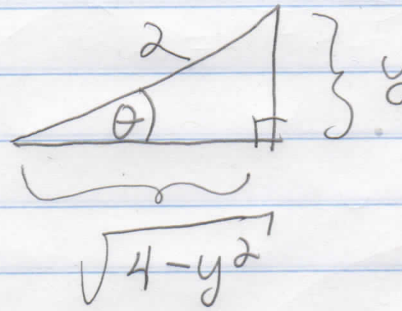
We need to show that

a) $\boxed{\delta(y) = \gamma(\phi(y))}$

b) $\phi'(y) > 0$, for $y \in (-2, 2)$.

a) Let $\theta = \overbrace{\arctan\left(\frac{y}{\sqrt{4-y^2}}\right)}^{\phi(y)}$ and

consider the triangle



Then $\cos(\theta) = \frac{\sqrt{4-y^2}}{2}$ and $\sin(\theta) = \frac{y}{2}$.

NOW compute

SO

$$\gamma(\phi(y)) = 2\underbrace{\cos(\phi(y))}_{\frac{\sqrt{4-y^2}}{2}} + i\,2\underbrace{\sin(\phi(y))}_{\frac{y}{2}} =$$

$$= \sqrt{4-y^2} + iy = \delta(y).$$

b) For $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\phi'(y) = \underbrace{\frac{1}{1+\left(\frac{y}{\sqrt{4-y^2}}\right)^2}}_{>0} \cdot \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{4-y^2}}\right)$.

The first factor is positive. It remains to show that the 2nd factor is positive as well.

$$\frac{\partial}{\partial y}\frac{y}{\sqrt{4-y^2}} = \frac{\partial}{\partial y}\, y\cdot(4-y^2)^{-\frac{1}{2}} =$$

$$= (4-y^2)^{-1/2} + y\cdot\left(-\tfrac{1}{2}\right)\cdot(4-y^2)^{-3/2}\cdot(-2y) =$$

$$= (4-y^2)^{-1/2} + y^2(4-y^2)^{-3/2} > 0.$$

Q.E.D