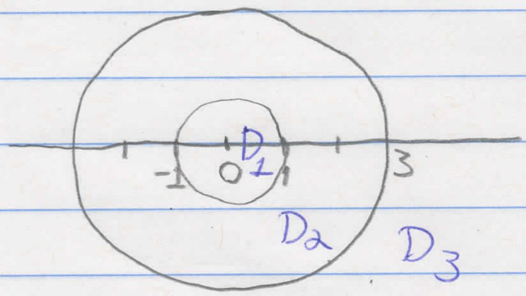
Sec 62 page 205: Extra Problem

$$f(z) = \frac{z}{(z-1)(z-3)}$$



Find the Laurent Series of $f(z)$ in each of the generalized annular domains centered at $z_0 = 0$.

Answer: $f(z) = \frac{A}{z-1} + \frac{B}{z-3} = \frac{A(z-3)+B(z-1)}{(z-1)(z-3)}$

$z = A(z-3) + B(z-1)$, so

$3 = 2B \Rightarrow B = 3/2$

$1 = -2A \Rightarrow A = -1/2$

$$f(z) = -\tfrac{1}{2} \cdot \frac{1}{z-1} + \tfrac{3}{2} \frac{1}{z-3}$$

Laurent Series of f in $D_1 := \{z : |z| < 1\}$:

$g(z) \overset{def}{:=} -\frac{1}{2}\frac{1}{z-1} = \frac{1}{2}\frac{1}{1-z} \underset{|z|<1}{=} \frac{1}{2}\sum_{n=0}^{\infty} z^n$ is absolutely convergent in D_1

$h(z) \overset{def}{:=} \tfrac{3}{2}\frac{1}{z-3} = \frac{-3}{2}\frac{1}{3-z} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{3}} \underset{\text{For } |z|<3}{=} -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$

So $\boxed{f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2} - \frac{1}{2\cdot 3^n}\right) z^n}$ for $\{z : |z| < 1\}$

Laurent Series of f in $D_2 := \{z ; 1<|z|<3\}$

$$g(z) = -\frac{1}{2}\,\frac{1}{z-1} = -\frac{1}{2z}\,\frac{1}{1-\left(\frac{1}{z}\right)} \underset{\substack{\text{For } |z|>1 \\ \text{so that } \left|\frac{1}{z}\right|<1}}{=} -\frac{1}{2z}\sum_{m=0}^{\infty}\left(\frac{1}{z}\right)^m =$$

$$= \sum_{m=0}^{\infty} -\frac{1}{2}\,\frac{1}{z^{m+1}} \underset{k=-1-m}{=} \sum_{k=-\infty}^{-1}\left(-\frac{1}{2}\right)z^k$$

$h(z) = \sum_{m=0}^{\infty} -\frac{1}{2\cdot 3^m}\, z^m$ as above is convergent in the annulus D_2 as well.

The Laurent series is thus

$$\boxed{f(z) = \sum_{m=-\infty}^{-1}\left(-\frac{1}{2}\right)z^m + \sum_{m=0}^{\infty}\frac{-1}{2\cdot 3^m}\, z^m}$$

Laurent Series of f in $D_3 := \{z ; |z|>3\}$:

The series for $g(z)$ is as in D_2.

$$h(z) = \frac{3}{2}\,\frac{1}{z-3} = \frac{3}{2z}\,\frac{1}{1-3/z} \underset{\substack{|z|>3 \text{ so that} \\ |3/z|<1}}{=} \frac{3}{2z}\cdot\sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^k =$$

$$= \sum_{k=0}^{\infty}\frac{3^{k+1}}{2}\, z^{-k-1} \underset{m=-k-1}{=} \sum_{m=-\infty}^{-1}\frac{3^m}{2}\, z^m$$

So, the Laurent series for f in D_3 is

$$\boxed{f(z) = \sum_{k=-\infty}^{-1}\left(-\frac{1}{2} + \frac{3^m}{2}\right)z^m}$$