Math 421 Midterm 2 Fall 2009

Name: My sol'n

1. (18 points) Compute the integral $\int_C \bar{z}\,dz$, where C is the triangle with vertices at the points 0, 1, and i, (traversed counterclockwise). *Caution: The integrand is the complex conjugate $\bar{z}$ of z.*

$C_1: \quad z(t) = t, \qquad 0 < t < 1$

$C_2: \quad z(t) = 1 + t(i-1) = 1-t+ti$

$C_3: \quad z(t) = i + t(-i) = (1-t)i$

$$\int_{C_1} \bar{z}\,dz = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\int_{C_2} \bar{z}\,dz = \int_0^1 [(1-t) - ti]\underbrace{\frac{dz}{dt}}_{(-1+i)}dt = \int_0^1 \overbrace{(t-1)+t}^{2t-1} + (\underbrace{t+1-t}_{1})i \; dt$$

$$= \left[(t^2 - t) + i(t)\right]_0^1 = i$$

$$\int_{C_3} \bar{z}\,dz = \int_0^1 -(1-t)i \cdot \underbrace{\frac{dz}{dt}}_{-i} dt = \int_0^1 \underbrace{-(1-t)}_{t-1}dt = \left[-t+\frac{t^2}{2}\right]_0^1 = -\frac{1}{2}$$

$$\int_C \bar{z}\,dz = \int_{C_1+C_2+C_3} \bar{z}\,dz = \frac{1}{2} + i + \left(-\frac{1}{2}\right) = \boxed{i}$$

<u>Method 2:</u> See problem 7 page 163 in section 49.

The integral $= 2i$ (Area of the triangle).

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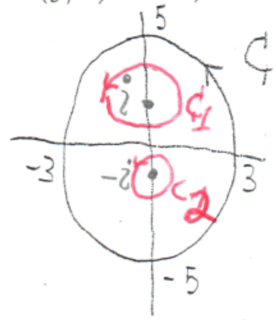
2. (18 points) Let C be the ellipse cut out by the equation $(x/3)^2 + (y/5)^2 = 1$, oriented counterclockwise. Compute $\displaystyle\int_C \frac{z^3\,dz}{(z-i)(z^2+1)}$.



$$I = \int_C \frac{z^3}{\underbrace{(z-i)(z-i)}_{(z-i)^2}(z+i)}\,dz =$$

$$= \int_{C_1+C_2} \frac{z^3}{(z-i)^2(z+i)}\,dz =$$

$$= \underbrace{\int_{C_1} \frac{z^3/z+i}{(z-i)^2}\,dz}_{=} + \underbrace{\int_{C_2} \frac{z^3/(z-i)^2}{z+i}\,dz}_{=}$$

Cauchy's Integral Formula

$$(2\pi i)\cdot \frac{d}{dz}\left(\frac{z^3}{z+i}\right)\Bigg|_{z=i} = \left(\frac{3z^2(z+i)-z^3(1)}{(z+i)^2}\right)\Bigg|_{z=i} = \frac{-3(2i)+i}{-4} = \frac{5}{4}i$$

$$-\frac{5}{2}\pi$$

$$(2\pi i)\cdot \underbrace{(-i)^3/(-i-i)^2}_{\frac{+i}{-4}} = +\frac{\pi}{2}$$

Answer:

$I = -2\pi$

3. (16 points) Suppose that $f(z)$ is entire and $|f(z)| \geq 1/2$, for all z in the complex plane. Prove that f is a constant function. Hint: The strategy is similar to the proof of the Fundamental Theorem of Algebra, but the actual proof is much simpler.

Liouiville's Theorem:

Let f be an entire function, such that $|f|$ is bounded from ABOVE by some positive real number M (i.e. $|f(z)| \leq M$ for all z in $\mathbb{C}$). Then f is a constant function.

Solution of the problem: (Compare with Problem 1 section 54 page 178).

Set $g(z) := \frac{1}{f(z)}$. Then $g(z)$ is entire, since $f(z)$ can never be zero, by assumption. Furthermore, $|g(z)| = \frac{1}{|f(z)|} \leq 2$ is bounded. Hence, by Liouiville's Theorem, $g(z) = c$, for some constant c.

Thus, $f(z) = \frac{1}{c}$ is constant as well. $\square$

4. (16 points) Let C be the unit circle parametrized by $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$.

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(a) Show that for all integers n, $\int_C (e^{(z^n)}/z)dz = 2\pi i$.

$$\int_C \frac{e^{(z^n)}}{z} dz \underset{\uparrow}{=} (2\pi i) \cdot \underbrace{e^{(0^n)}}_{1} = 2\pi i$$

Cauchy's Integral Formula with

$f(z) = e^{(z^n)}$ (analytic on the closed unit disk)

$z_0 = 0$

(b) Derive the integration formula $\int_0^{2\pi} e^{\cos(n\theta)} \cos(\sin(n\theta))d\theta = 2\pi$, for every integer n.

$$2\pi i = \int_C \frac{e^{(z^n)}}{z} dz = \int_0^{2\pi} \frac{e^{((e^{i\theta})^n)}}{e^{i\theta}} \cdot i e^{i\theta} d\theta =$$

$$= \int_0^{2\pi} e^{(e^{in\theta})} \cdot i \, d\theta = i \int_0^{2\pi} e^{[\cos(n\theta) + i \sin(n\theta)]} d\theta =$$

$$= i \int_0^{2\pi} e^{\cos(n\theta)} \cdot \left[\cos(\sin(n\theta)) + i \sin(\sin(n\theta))\right] d\theta,$$

Equating the imaginary part of both sides, we get the desired equation.

$(0, 2\pi)$
$\downarrow$
$\arg(z)$

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5. (16 points) Let the domain D be the complex plane minus the non-negative part of the x-axis. Let $\log(z)$ be the branch of the logarithm function with argument in the interval $(0, 2\pi)$, so that $log(z)$ is analytic in D. Set $f(z) := e^{(1/2)\cdot\log(z)}$. Note that $f(z)$ is a branch of the multi-valued function $\sqrt{z}$.

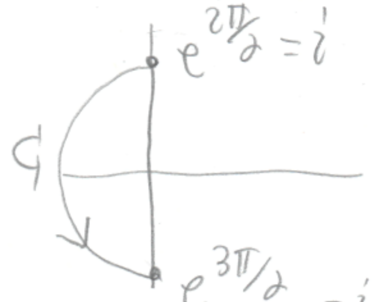
(a) Find a single valued anti-derivative $F(z)$ of $f(z)$ in D. Express your answer in terms of the above branch of $\log(z)$ and avoid using multi-valued rational powers of z. Check that your answer is indeed an anti-derivative, by explicitly differentiating it.

$$F(z) = \frac{2}{3} e^{3/2 \log(z)} = \frac{2}{3} e^{1/2 \log(z)} \cdot z. \quad \text{Note } F(z) = \frac{2}{3} e^{\left[\frac{3\ln|z|}{2} + \frac{3i\arg(z)}{2}\right]}$$

$$\frac{\partial F}{\partial z} = \frac{2}{3} \cdot \frac{3}{2} e^{3/2 \log(z)} \cdot \frac{1}{z} = e^{3/2 \log(z)} \cdot \frac{1}{z} = e^{1/2 \log(z)} \cdot z \cdot \frac{1}{z} =$$
$$= e^{1/2 \log(z)}$$

(b) Let C be the contour $z(\theta) = e^{i\theta}$, $\pi/2 \le \theta \le 3\pi/2$. Prove the equality

$$\int_C f(z)dz = \frac{2\sqrt{2}}{3}.$$



$$\int_C f(z)dz = F(e^{i\frac{3\pi}{2}}) - F(e^{i\frac{\pi}{2}}) =$$

$$= \frac{2}{3}\left[e^{0 + \underbrace{\frac{3}{2} i\left(\frac{3\pi}{2}\right)}_{9\pi i/4}} - e^{0 + \underbrace{\frac{3}{2} i\left(\frac{\pi}{2}\right)}_{3\pi i/4}}\right] =$$

$$= \frac{2}{3}\left[e^{\frac{\pi i}{4}} - e^{\frac{3\pi i}{4}}\right] =$$

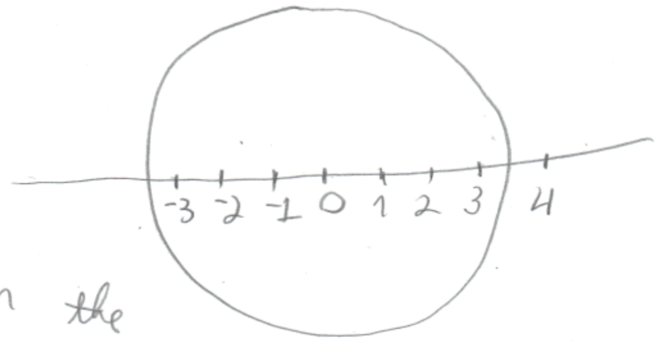
$$= \frac{2}{3}\left[\frac{1}{\sqrt{2}}(1+i) - \frac{1}{\sqrt{2}}(-1+i)\right] = \frac{2}{3\sqrt{2}} \cdot 2 = \frac{2\sqrt{2}}{3}$$

$\frac{1}{\sqrt{2}}(-1+i) = e^{3\pi i/4}$

$e^{\pi i/4} = \frac{1}{\sqrt{2}}(1+i)$

$f(z) = z^5 + 3z + 7$

6. (16 points) Let C be a circle of radius $7/2$ centered at the origin oriented counter-clockwise. Set $g(n) := \int_C \frac{z^5+3z+7}{(z-n)^3}\,dz$. Compute $g(n)$ for all integers n. Justify your answer!!!



If $m > 4$ or $m < -4$, then $\frac{z^5+3z+7}{(z-m)^3}$ is analytic on the contour C and in the disk bounded by it, so

$$g(m) = \int_C \frac{z^5+3z+2}{(z-m)^3}\,dz = 0,$$ by Cauchy-Goursat,

If $-3 \leq m \leq 3$, then by Cauchy's Integral Formula

$$f''(m) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-m)^3}\,dz = \frac{1}{\pi i}\, g(m)$$

So $$g(m) = \pi i \cdot \underbrace{\frac{\partial^2}{\partial z^2}\left(z^5+3z+7\right)\Big|_{z=m}}_{5\cdot 4\cdot z^3\big|_{z=m}} = \boxed{20\pi i\, m^3}$$