Math 421 Midterm 2 Solution Fall 2002

Solve the first 5 problems and only **one** out of problems 6 and 7.

1. (20 points)

   Compute the contour integral
$$\int_C e^{\bar{z}} dz,$$
   where C is the boundary of the rectangle with vertices at the points 0, 2, $2+i$, and i, oriented counterclockwise. *Caution: the exponent of the integrand is the complex conjugate* $\bar{z}$ *of* z.

   Answer: Label the edges of the rectangle by C_1 (bottom), C_2 (right), C_3 (top), and C_4 (left). The edges are parametrized as follows:

$$\begin{array}{llll} C_1: & z(t) = 2t, & 0 \le t \le 1, \text{ so} & e^{\overline{z(t)}}dz = e^{2t} \cdot 2dt, \\ C_2: & z(t) = 2+it, & 0 \le t \le 1, \text{ so} & e^{\overline{z(t)}}dz = e^{2-it} \cdot idt, \\ C_3: & z(t) = 2+i-2t, & 0 \le t \le 1, \text{ so} & e^{\overline{z(t)}}dz = e^{2-2t-i} \cdot (-2)dt, \\ C_4: & z(t) = i-it, & 0 \le t \le 1, \text{ so} & e^{\overline{z(t)}}dz = e^{i(t-1)} \cdot (-i)dt. \end{array}$$

$$\int_C e^{\bar{z}}dz = \int_0^1 e^{2t} \cdot 2dt + \int_0^1 e^{2-it} \cdot idt + \int_0^1 e^{2-2t-i} \cdot (-2)dt + \int_0^1 e^{i(t-1)} \cdot (-i)dt =$$
$$[e^2 - 1] - [e^{2-i} - e^2] + [e^{-i} - e^{2-i}] - [1 - e^{-i}] = -2 + 2e^2 - 2e^{2-i} + 2e^{-i} =$$
$$-2 + 2e^2 - 2e^2(\cos(1) - i\sin(1)) + 2(\cos(1) - i\sin(1)).$$

2. (18 points) Let C be the circle of radius 5 centered at the origin and transversed counterclockwise. Compute
$$\int_C \frac{e^z}{z^2+1}dz. \tag{1}$$

   Answer: Factoring $z^2 + 1 = (z-i)(z+i)$, we see that the function $\frac{e^z}{z^2+1}$ is analytic on the region $\mathbb{C} \setminus \{i, -i\}$. This region is *not* simply connected. Let C_1 and C_2 be the circles with radius 1/2, oriented counterclockwise, and centered at i and $-i$ respectively. The generalization of Cauchy-Goursat's Theorem, for non-simply-connected regions, implies the equality:
$$\int_C \frac{e^z}{z^2+1}dz = \int_{C_1} \frac{e^z}{z^2+1}dz + \int_{C_2} \frac{e^z}{z^2+1}dz.$$

   Cauchy's Theorem implies the equalities
$$\int_{C_1} \frac{\left(\frac{e^z}{z+i}\right)}{z-i}dz = 2\pi i \frac{e^i}{2i},$$
$$\int_{C_2} \frac{\left(\frac{e^z}{z-i}\right)}{z+i}dz = 2\pi i \frac{e^{-i}}{-2i}.$$

   Thus, the integral (1) is equal to $\pi[e^i - e^{-i}] = 2\pi \sin(1) i$.

3. (18 points) Let C be the circle $\{z \text{ such that } |z| = 10\}$, transversed counterclockwise. Evaluate

$$\int_C \frac{\sin(z)}{(z-\frac{\pi}{2})^n} dz,$$

for all integers n (positive, negative, or zero).

Answer: If $n \leq 0$, then the function $\frac{\sin(z)}{(z-\frac{\pi}{2})^n}$ is entire, and the integral is equal to 0, by Cauchy-Goursat's Theorem.

For $n \geq 1$, Cauchy's Formula yields

$$\frac{(n-1)!}{2\pi i}\int_C \frac{\sin(z)}{(z-\frac{\pi}{2})^n} dz = \sin^{(n-1)}\left(\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ (-1)^k & \text{if} \quad n = 2k+1. \end{cases}$$

Summarizing, we get

$$\int_C \frac{\sin(z)}{(z-\frac{\pi}{2})^n} dz = \begin{cases} 0 & \text{if } \ n \leq 0, \\ 0 & \text{if } \ n \text{ is even} \\ (-1)^k \frac{2\pi i}{(n-1)!} & \text{if } \ n = 2k+1 > 0. \end{cases}$$

4. (16 points) Prove the equality

$$\frac{\pi}{3} \leq \left|\int_C \frac{z+1}{z-1} dz\right| \leq 3\pi, \tag{2}$$

where C is the semi-circle, given by the parametrization $z(t) = 2e^{it}$, $0 \leq t \leq \pi$.

Answer: Both inequalities happen to hold. The lower bound is not as straightforward as I intended, and the upper bound on the right was ment to be 6π. Nevertheless, this question happened to be the easiest for most students. Full credit was given, if the method for establishing the right hand inequality was exhibited. (Most students did get full credit here). We find an upper bound for the ingtegral as follows.

Let M be an upper bound for $\frac{z+1}{z-1}$ over the circle C. Then

$$\left|\int_C \frac{z+1}{z-1} dz\right| \leq \int_C \left|\frac{z+1}{z-1}\right| |dz| \leq \int_C M|dz| = M \cdot \text{length}(C) = 2\pi M.$$

Now, the upper bound $M = 3$ for the integrand is found using the inequalities

$$\begin{aligned} |z+1| &\leq |z| + 1 = 3, \\ |z-1| &\geq |z| - 1 = 1, \\ \left|\frac{z+1}{z-1}\right| &\leq 3. \end{aligned}$$

We can easily evaluate the integral explicitly, and verify that both inequalities in (2) hold (this was not needed to get full credit). Observe the equality

$$\frac{z+1}{z-1} = 1 + \frac{2}{z-1}.$$

Denote by $C-1$ the translate of the semi-circle one unit to the left. Then $C-1$ is in the domain of the branch of $\log(z)$ with argument in $(-\frac{\pi}{2}, \frac{3\pi}{2})$. So,

$$\int_C \frac{z+1}{z-1}dz = \int_C [1+\frac{2}{z-1}]dz = [z+2\log(z-1))]_2^{-2} =$$
$$-4+2[\log(-3)-\log(1)] = -4+2[(\ln(3)+i\pi)-0] =$$
$$2\ln(3)-4+2\pi i.$$

5. (16 points) Let C_1 be the curve, consisting of the piece of the graph of $y=\sin(x)$, given by the parametrization

$$z(x) = (x+i\sin(x)), \quad 0 \le x \le 2\pi.$$

Let C_2 be the piece of the graph of $y=-\sin(x)$, given by the parametrization

$$z(x) = (x-i\sin(x)), \quad 0 \le x \le 2\pi.$$

Compute the difference

$$\int_{C_1} \frac{dz}{z-\frac{\pi}{2}} - \int_{C_2} \frac{dz}{z-\frac{\pi}{2}}$$

Answer: The contours C_1 and C_2 have the same end points. Thus, $C_1 - C_2$ is a closed contour. In fact, $C_1 - C_2$ is the figure ∞, whose left loop encloses $\frac{\pi}{2}$. Denote the left loop of $C_1 - C_2$ by Γ_1 and the right loop by Γ_2. Then Γ_1 is oriented clockwise, Γ_2 counterclockwise, and both are simple closed curves. The integral over Γ_2 vanishes, because $\frac{1}{z-\frac{\pi}{2}}$ is analytic on Γ_2 and on the domain it encloses. The integral over $-\Gamma_1$ (with the counterclockwise orientation) is $2\pi i$, by Cauchy's formula. We get

$$\int_{C_1} \frac{dz}{z-\frac{\pi}{2}} - \int_{C_2} \frac{dz}{z-\frac{\pi}{2}} = \int_{\Gamma_1} \frac{dz}{z-\frac{\pi}{2}} + \int_{\Gamma_2} \frac{dz}{z-\frac{\pi}{2}} = -2\pi i + 0 = -2\pi i.$$

6. (12 points) Determine whether the following statements are true or false. **Justify your answers**.

a) Let C be any contour from 1 to $8i$, which does not pass through 0. Then the following equality holds

$$\int_C \frac{dz}{z} = \ln(8) + \frac{\pi}{2}i.$$

Answer: The statement is **FALSE**. The integral depends on the contour chosen. If we choose C_1 to be the straight line from 1 to $8i$, the above equality holds. Choose, for example, C_2 to be another contour from 1 to $8i$, passing "below" 0, so that $C_1 - C_2$ is a simple closed contour enclosing 0. Then

$$\int_{C_2} \frac{dz}{z} = \int_{C_1} \frac{dz}{z} - \int_{C_1 - C_2} \frac{dz}{z} = \int_{C_1} \frac{dz}{z} - 2\pi i.$$

b) Let $P(z) = (z-1)(z-2i)(z+4+5i)(z-9i)$ and C_R the circle of radius R, centered at the origin, transversed counterclockwise. Let I_R be the value of the integral

$$\int_{C_R} \frac{dz}{P(z)}.$$

Then $I_R = I_{100}$, for all radii R satisfying $R \geq 100$.

Answer: The statement is **TRUE**. All the roots of $P(z)$ are contained in the closed disk $\{z \text{ such that } |z| \leq 9\}$. The function $\frac{1}{P(z)}$ is analytic outside this disk. The statement follows from the *Principle of Deformation of Paths* (Corollary 2 page 118 in the text).

7. (12 points) a) (4 points) Let $f(z)$ and $g(z)$ be entire functions, and set $P(z) := f(z)g(z)$. Prove the equality

$$\frac{P'(z)}{P(z)} \;=\; \frac{f'(z)}{f(z)} + \frac{g'(z)}{g(z)}.$$

Answer:

$$\frac{P'}{P} \;=\; \frac{f'g + fg'}{fg} \;=\; \frac{f'}{f} + \frac{g'}{g}.$$

b) (8 points) Let $P(z)$ be a polynomial of degree n. Let C_R the circle of radius $R > 0$, centered at the origin, transversed counterclockwise. Prove, the equality

$$\int_{C_R} \frac{P'(z)}{P(z)} dz \;=\; 2n\pi i,$$

provided R is sufficiently large. *Hint: Do the case $n = 1$ first.*

Proof: The Fundamental Theorem of Algebra implies, that $P(z)$ factors as a product of linear terms

$$P(z) \;=\; c \cdot (z - \lambda_1) \cdot (z - \lambda_2) \cdot \cdots \cdot (z - \lambda_n).$$

Using part a) $n + 1$ times, we get

$$\frac{P'}{P} \;=\; 0 + \frac{1}{z - \lambda_1} + \cdots + \frac{1}{z - \lambda_n}.$$

Set R_0 to be the maximum of all the absolute values $|\lambda_i|$, $1 \leq i \leq n$. The circle C_R encloses all the roots of P, if $R > R_0$. For such a radius, we get

$$\int_{C_R} \frac{P'(z)}{P(z)} dz \;=\; \sum_{i=1}^{n} \int_{C_R} \frac{dz}{z - \lambda_i} \;=\; \sum_{i=1}^{n} 2\pi i \;=\; 2n\pi i.$$

The second equality is established using Cauchy's Formula.