Math 421 Midterm 2 Fall 2002

Solve the first 5 problems and only **one** out of problems 6 and 7.

1. (20 points)

   Compute the contour integral
   $$\int_C e^{\bar{z}} dz,$$
   where C is the boundary of the rectangle with vertices at the points 0, 2, $2+i$, and i, oriented counterclockwise. *Caution: the exponent of the integrand is the complex conjugate $\bar{z}$ of z.*

2. (18 points) Let C be the circle of radius 5 centered at the origin and transversed counterclockwise. Compute
   $$\int_C \frac{e^z}{z^2+1} dz.$$

3. (18 points) Let C be the circle $\{z \text{ such that } |z| = 10\}$, transversed counterclockwise. Evaluate
   $$\int_C \frac{\sin(z)}{(z-\frac{\pi}{2})^n} dz,$$
   for all integers n (positive, negative, or zero).

4. (16 points) Prove the equality
   $$\left| \int_C \frac{z+1}{z-1} dz \right| \leq 6\pi,$$
   where C is the semi-circle, given by the parametrization $z(t) = 2e^{it}$, $0 \leq t \leq \pi$.

5. (16 points) Let C_1 be the curve, consisting of the piece of the graph of $y = \sin(x)$, given by the parametrization
   $$z(x) = (x + i\sin(x)), \quad 0 \leq x \leq 2\pi.$$
   Let C_2 be the piece of the graph of $y = -\sin(x)$, given by the parametrization
   $$z(x) = (x - i\sin(x)), \quad 0 \leq x \leq 2\pi.$$
   Compute the difference
   $$\int_{C_1} \frac{dz}{z-\frac{\pi}{2}} - \int_{C_2} \frac{dz}{z-\frac{\pi}{2}}$$

6. (12 points) Determine whether the following statements are true or false. **Justify your answers**.

a) Let C be any contour from 1 to $8i$, which does not pass through 0. Then the following equality holds

$$\int_C \frac{dz}{z} \;=\; \ln(8) + \frac{\pi}{2} i.$$

b) Let $P(z) = (z-1)(z-2i)(z+4+5i)(z-9i)$ and C_R the circle of radius R, centered at the origin, transversed counterclockwise. Let I_R be the value of the integral

$$\int_{C_R} \frac{dz}{P(z)}.$$

Then $I_R = I_{100}$, for all radii R satisfying $R \geq 100$.

7. (12 points) a) (4 points) Let $f(z)$ and $g(z)$ be entire functions, and set $P(z) := f(z)g(z)$. Prove the equality

$$\frac{P'(z)}{P(z)} \;=\; \frac{f'(z)}{f(z)} + \frac{g'(z)}{g(z)}.$$

b) (8 points) Let $P(z)$ be a polynomial of degree n. Let C_R the circle of radius $R > 0$, centered at the origin, transversed counterclockwise. Prove, the equality

$$\int_{C_R} \frac{P'(z)}{P(z)} dz \;=\; 2n\pi i,$$

provided R is sufficiently large. *Hint: Do the case* $n = 1$ *first.*