Notation:
$\mathbb{N}$ is $\mathbb{P}$ in our textbook
$\mathcal{P}(X)$ = the set of all subsets of $X$.

**MATH 300.1 – Fall 2011**
**Homework Set # 10**

These problems will not be collected but they are essential practice for the final exam. They will be discussed in the remaining Discussion Sections.

A different textbook was used that semester: Beck and Geoghegan, "The art of proof."

**Problem 1.** Prove that $\sqrt[3]{4}$ is irrational.

**Problem 2.** Prove that if $a$ is rational and $b$ is irrational then $a+b$ is irrational.

**Problem 3.** Prove that $\sqrt{2}+\sqrt{3}$ is irrational. Can you generalize this result?

**Problem 4.** Prove that $\log_{10} 2$ is irrational.

**Problem 5.** Consider the relation $R$ on $\mathbb{R}$ defined by:

$$R \doteq \{(x,y) \in \mathbb{R}\times\mathbb{R} : x-y \in \mathbb{Q}\}.$$

Prove that $R$ is an equivalence relation.

(is either finite, or i.e. has the same cardinality as the set $\mathbb{P}$ of positive integers)

**Problem 6.** Show that if $Y$ is a countable set, and $f : X \to Y$ is injective, then $X$ is countable.

**Problem 7.** Prove that a set $A$ is countable if and only if there exists an injective map $f : A \to \mathbb{N}$. This was proven in class.

**Problem 8.** Suppose $X$ is finite and $Y$ is countably infinite. Let $f\colon Y \to \mathbb{N}$ be a bijection. Use $f$ to define an explicit bijection

i.e. $\#Y = \#\mathbb{P}$

$$F\colon X\times Y \to \mathbb{N}.$$

denoted $\mathbb{P}$ in our textbook

**Problem 9.** Let $A_1,\ldots,A_n$ be countably infinite sets. Prove that

$$A_1\cup\cdots\cup A_n$$

is countably infinite.

**Problem 10.** Let $\Delta = \{(m,n)\in\mathbb{N}\times\mathbb{N} : m\geq n\}$.

a) Draw a picture of $\Delta$.

b) Prove that the map $F\colon \mathbb{N}\times\mathbb{N}\to\Delta$ defined by

$$F(a,b) = (a+b-1,b)$$

is a bijection.

c) Let $G\colon \Delta\to\mathbb{N}$ be the map

$$G(m,n) = \frac{m(m-1)}{2}+n.$$

Compute $G(1,1)$, $G(2,1)$, $G(2,2)$, $G(3,1)$, $G(3,2)$, $G(3,3)$, $G(4,1)$.

d) Prove that $G$ is a bijection.

**e)** Compute the composition $H = G \circ F$ to obtain an explicit bijection
$$H\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$

**Problem 11.** Given that $\mathbb{N} \times \mathbb{N}$ is countable (as proved in class), show by induction that the Cartesian product
$$\underbrace{\mathbb{N} \times \mathbb{N} \times \dots \mathbb{N}}_{k}$$
of $\mathbb{N}$ with itself $k$ times is countable, for any $k \in \mathbb{N}$.

**Problem 12.** Let $X$ be the set of *finite* subsets of $\mathbb{N}$. Show that $X$ is countable. Hint: it suffices to find an injection from $X$ into $\mathbb{N}$; how about $f(A) = \prod_{n \in A} p_n$ where $p_n$ is the $n$th prime number?

The set of all subsets of $\mathbb{N}$

**Problem 13.** Find out what's bogus about the following "proof" that $\mathcal{P}(\mathbb{N})$ is countable.
**Claim.** There is an injective map from $\mathcal{P}(\mathbb{N})$ into $\mathbb{N}$ hence, $\mathcal{P}(\mathbb{N})$ is countable.
**Proof.** If $A$ is a subset of $\mathbb{N}$, define $f(A) = \prod_{n \in A} p_n$ where $p_n$ is the $nth$ prime number. Then $f$ is injective. Done.

**Problem 14.** Let $I = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers. Prove that $I$ is uncountable.

**Problem 15.** Give a bijection from $(0,1) = \{x \in \mathbb{R} | 0 < x < 1\}$ to $\mathbb{R}$, thereby showing that $|(0,1)| = |\mathbb{R}|$. Hint: think about a function that has an asymptote going to $-\infty$ near 0 and one going to $+\infty$ near 1.

**Problem 16.** Prove Proposition 13.29 from the text. (Hint: Find a nice function that maps $(a,b)$ to $(c,d)$ and sends $a$ to $c$ and $b$ to $d$. If your function is "nice" enough, it will be bijective.)

**Problem 17.** Let $f\colon \mathbb{R} \to \mathbb{N}$ be a surjective function and define an equivalence relation on $\mathbb{R}$ by $x \sim y \Leftrightarrow f(x) = f(y)$. Prove that there exists $x \in \mathbb{R}$ such that $[x]$ is uncountable.

**Problem 18.** Suppose $X$ is a non-empty set and let $f : X \to \mathcal{P}(X)$ be defined by
$$f(x) = X \setminus x.$$
Consider the subset $A = \{x \in X | x \notin f(x)\}$ of $X$ (which plays a prominent role in Cantor's theorem). Determine $A$ for the particular $f$ we have just defined.

**Problem 19.** Consider the set

set of all functions from $\mathbb{N}$ to $\{0,1\}$
$$X = \text{Maps}(\mathbb{N}, \{0,1\}) = \{f : \mathbb{N} \to \{0,1\}\};$$
i.e, $X$ is the set of sequences consisting of 0's and 1's. As usual, we identify the map $f$ with the sequence $a_n = f(n)$. We say that two sequences $\{a_n\}$ and $\{b_n\}$ have the *same tail* if and only if there exists $n_0 \in \mathbb{N}$ such that $a_n = b_n$ for all $n \geq n_0$.

**a)** Prove that "having the same tail" is an equivalence relation.
**b)** Prove that for every $\{a_n\} \in X$ the equivalence class $[\{a_n\}]$ is a countable set.
**c)** Prove that there are uncountably many equivalence classes in $X$. Hint: Use contradiction, together with the fact, proved in class, that $X$ is uncountable.

**Problem 20.** Prove all Propositions from Chapter 13 not proved in class.