Math 300.2 Quiz 5 Spring 2016

Name: Solution

Let $f : X \to Y$ and $g : Y \to X$ be functions such that $g \circ f = 1_X$, where $1_X : X \to X$ is the identity function sending each $x \in X$ to itself. (*)

1. (70 points) Prove that f is injective and g is surjective.

Def: f is injective, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, for all $x_1, x_2 \in X$.

Proof that f is injective:

Assume that $f(x_1) = f(x_2)$. Then

$$\underset{\text{by (*)}}{x_1 =} g(f(x_1)) = g(f(x_2)) \underset{\text{by (*)}}{=} x_2.$$

Hence, $x_1 = x_2$ and f is injective.

Proof that g is surjective. (Def: g is surjective if every $a \in X$ belongs to the image of g.)

Let $a \in X$. Then $a = g(f(a))$, by the equality $g \circ f = 1_X$ above. Hence, a belongs to the image of g, and g is surjective.

See back for the second part of the problem

2. (30 points) Show, via an example, that f and g need not be bijective.

Example 1:

Let X be the set consisting of the single element 1, $Y = \mathbb{Z}$,

$f : \{1\} \to \mathbb{Z}$, given by $f(1) = 1$,

and $g : \mathbb{Z} \to \{1\}$, the constant map sending every $n \in \mathbb{Z}$ to 1. Then

$g \circ f = 1_X$, but f is not surjective

and g is not injective.

Example 2:

Let $X = [0, \infty)$, $Y = \mathbb{R}$,

$f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$,

$g : \mathbb{R} \to [0, \infty)$, $g(y) = y^2$,

then f is injective, but not surjective, and g is surjective, but not injective.

Nevertheless, $g(f(x)) = (\sqrt{x})^2 = x$, for all $x \in [0, \infty)$.