Chapter 3: Review Questions for Midterm 2

1. Solve the linear congruences:
(a) $4x \equiv 6 \pmod{14}$ (b) $29x \equiv 62 \pmod{128}$.
2. Construct the addition and multiplication tables for $\mathbb{Z}_6$.
3. Suppose the 9-digit number $1234x6789$ is divisible by 9. Find all possible values of x.
4. Solve for $x \in \mathbb{Z}$: $x^6 \equiv 6x \pmod{7}$.
5. Find the multiplicative inverse of $[3]$ in $\mathbb{Z}_{41}$.
6. Find the remainder when 14^{181} is divided by 99.
7. If $2p^2 = q^3$, where $p, q \in \mathbb{Z}$, show that 2 is a common divisor of p and q.
8. Show that an integer of the form $7m + 5$ can not be a perfect square.
9. Show that $\sqrt{6}$ is not a rational number.

The Chinese Remainder Theorem.

10. Solve the simultaneous congruences:
(i) $x \equiv 5 \pmod{7}$, $x \equiv 23 \pmod{25}$.
(ii) $2x \equiv 11 \pmod{13}$, $3x \equiv 7 \pmod{8}$, $7x \equiv 5 \pmod{9}$.
11. Find the last two digits of 556^{3333}.
12. Show that if p, q are integers, not divisible by 3 or 5, then $p^4 \equiv q^4 \pmod{15}$.
13. Solve for $x \in \mathbb{Z}$: $x^2 \equiv 4 \pmod{30}$.
14. Solve for $x \in \mathbb{Z}$: $x^{32} + x + 1 \equiv 0 \pmod{35}$.

Relations and Equivalence Relations.

15. Determine whether the following relations on $\mathbb{Z}$ are reflexive, symmetric, or transitive.
(a) aRb if and only if $a + b \neq 1$.
(b) aRb if and only if $a - b \geq 0$.
(c) aRb if and only if $a \neq b$.
16. Show that the following relation is an equivalence relation: for any real numbers a and b, aRb if and only if $a - b = 2\pi k$ where $k \in \mathbb{Z}$.

Chapter 4:

12. Prove using Mathematical Induction: $\forall n \in \mathbb{P}$,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \geq 2 - \frac{1}{n}.$$

13. Prove using Mathematical Induction: Let $x \neq -1$. $\forall n \in \mathbb{P}$,

$$1 - x + x^2 + \cdots + (-1)^n x^n = \frac{1 - (-x)^{n+1}}{1 + x}.$$

14. A sequence of integers $x_1, x_2, x_3, \cdots$, is defined by $x_1 = 1, x_2 = 5$ and the recursion

$$x_n = 5x_{n-1} - 6x_{n-2}, \forall n \geq 3.$$

Find an expression for x_n and use Mathematical Induction to prove that the expression is correct.

15. Find an expression for

$$S_n = 1 - 3 + 5 - 7 + \cdots (-1)^{n+1}(2n - 1), \text{ where } n = 1, 2, 3, \cdots,$$

and prove the expression for S_n is correct.

Questions from Spring 2014 midterm: (Sol'n is on the web).

7. (15 points) Use induction to prove that, for every natural number n,

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$$

8. (15 points) Prove that $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^{2k}=3^n$. (Hint: Use the Binomial Theorem.)

9. (10 points) Let n be a natural number. Prove that if n^2 is divided by 4, the remainder is either 0 or 1.

- Second Midterm: Wednesday, March 30, 6:00 to 7:30 PM. Room: LGRC A301.
  - Review for second midterm: Monday, March 28, from 3:00 to 4:30 PM, Room: LGRT 204.
- You may bring one 8.5" x 11" sheet of **notes** (both sides) to all exams.
- Material for midterm 2: Ch 4, and Ch 3 (up to Sec 3.6 (including))