Math 300 Section 2 Midterm 2 Spring 2016

Name:__________________________

1. (15 points) Define the sequence $x_n$ as follows. $x_1 = 1$, $x_2 = 5$, and

$$x_n = x_{n-1} + 2x_{n-2}, \text{ for all } n \geq 3.$$

Find an expression for $x_n$ and prove, by induction, that the expression is correct. Hint: Compute the difference $x_n - 2^n$ for the first few terms.

2. (15 points) Let $p$ be a prime.
   (a) Prove that $\binom{p}{k} \equiv 0 \pmod{p}$, for $0 < k < p$.
   (b) Use induction on $n$ to prove that $n^p \equiv n \pmod{p}$, for all positive integers $n$ (Fermat's Little Theorem). Credit will be given only for a proof by induction. Hint: In the induction step you will need the Binomial Theorem and part 2a.

3. (15 points) Determine the number of congruence classes which solve the linear congruence $9x \equiv 6 \pmod{15}$ and find all of them. Justify your answer!

4. (15 points) Find all integers $x$ solving the congruence $x^{12} \equiv 5x \pmod{13}$. Justify your answer.

5. (15 points) Find the inverse of $[25]$ in $\mathbb{Z}_{41}$.

6. (15 points) Find all integers $x$ solving the simultaneous congruences

$$x \equiv 2 \pmod{5}, \quad (1)$$
$$x \equiv 18 \pmod{24}. \quad (2)$$

Justify your answer.

7. (15 points) Let $R$ be the relation on the integers $\mathbb{Z}$ given by $xRy$ if and only if $x^2 + x + 1 \equiv y^2 + y + 1 \pmod{5}$.
   (a) Show that $R$ is an equivalence relation.
   (b) Find a pair of integers $(x, y)$, such that $x \not\equiv y \pmod{5}$ and $xRy$.
   (c) The equivalence relation $R$ determines a partition of $\mathbb{Z}$ into equivalence classes with respect to $R$. Determine the number of equivalence classes in this partition and find a representative for each equivalence class. Justify your answer!