

Name: Solution

1. (15 points) Let P and Q be statements. Find the truth tables of the following statements and use them to determine if they are equivalent.

 - $P \Rightarrow Q$.
 - $(\text{NOT } Q) \Rightarrow (\text{NOT } P)$.

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$(\neg Q) \Rightarrow (\neg P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

2. (10 points) Let the universe of discourse be the real numbers. Prove or give a counter example to the following statement.

$$\forall x \exists y (x^2 > y^2).$$

"For all x there exists y , such that $x^2 > y^2$."

Counter example; $x=0$, There does not exist a real number y , such that $0^2 > y^2$.

3. (10 points) Let the universe of discourse be the real numbers. Write the contrapositive of the following statement.
If $x < -2$, then $x^2 > 4$.

If $x^2 \leq 4$, then $x \geq -2$.

4. (15 points) Let S and T be sets. Use the contrapositive method to prove the following statement.

$$(S \cap T = \emptyset) \text{ AND } (S \cup T = T) \Rightarrow (S = \emptyset).$$

Contrapositive (equivalent statements):

If $S \neq \emptyset$, then $((S \cap T \neq \emptyset) \text{ OR } (S \cup T \neq T))$.

Proof: If $S \neq \emptyset$, then there exists an element $x \in S$.

If $x \in T$, then $x \in S \cap T$, and so $S \cap T \neq \emptyset$,

If $x \notin T$, then $S \cup T \neq T$, since $x \in S \cup T$.

Hence $(S \cap T \neq \emptyset) \text{ OR } (S \cup T \neq T)$ is true.

QED

5. (10 points) How many positive divisors does 360^3 have? Justify your answer!

$$360 = 10 \cdot 6^2 = 2^3 \cdot 3^2 \cdot 5$$

$360^3 = 2^9 \cdot 3^6 \cdot 5^3$. The divisors of 360^3 are all of the form $2^i 3^j 5^k$, where $0 \leq i \leq 9$, $0 \leq j \leq 6$, $0 \leq k \leq 3$.
possibilities possib. possib.

We thus have $10 \cdot 7 \cdot 4 = 280$ divisors of 360^3 .

6. (10 points) Let x, y, z , and w be positive integers. Assume that $\gcd(x, z) = 1$ and that $y|z$. Prove that if $x|yw$, then $x|w$.

Step 1:

We first show that $\gcd(x, y) = 1$. Indeed, let c be a positive common divisor of x and y . Then $c|y$ and $y|z \Rightarrow c|z$. So c is a positive common divisor of x and z . Hence $c=1$, since $\gcd(x, z) = 1$, by assumption.

Step 2: If $x|yw$, then $x|w$, by Prop 2.28,
since $\gcd(x, y) = 1$. QED

$$33x + 18y = \boxed{r}$$

7. (15 points) a) Use the Extended Euclidean Algorithm to find a particular solution of the equation $33x + 18y = \gcd(33, 18)$.

6 pts

x	y	r	g
1	0	33	
0	1	18	
1	-1	15	1
-1	2	3	1
		0	

$$\gcd(33, 18) = 3$$

$$33(-1) + 18 \cdot (2) = 3$$

$$50 \cdot 3$$

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- b) Find all the integer solutions of the equation $33x + 18y = 150$. Show all your work.

5 pts

$x_0 = 50 \cdot (-1) = -50$, $y_0 = 50 \cdot 2 = 100$ is a particular solution. The general sol'm is

$$\begin{aligned} x &= -50 + \frac{18}{3}m = -50 + 6m \\ y &= 100 - \frac{33}{3}m = 100 - 11m \end{aligned} \quad \left\{ \text{general sol'n.} \right.$$

- c) Find all positive integer solutions of the equation $33x + 18y = 150$.

4 pts

$$0 < x = -50 + 6m \Leftrightarrow 6m > 50 \Leftrightarrow m > \frac{50}{6} = 8\frac{1}{3}$$

$$0 < y = 100 - 11m \Leftrightarrow 11m < 100 \Leftrightarrow m < \frac{100}{11} = 9\frac{1}{11}$$

$$\text{so } \boxed{m = 9}.$$

The unique solution is

$$\boxed{x = 4, y = 1.}$$

5 pts

8. (15 points) a) Let x be a positive integer and $x = p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n}$ its prime decomposition. What can you say about the parity (even or odd) of the exponents in the prime decomposition of x^2 ?

$$x^2 = p_1^{2d_1} p_2^{2d_2} \cdots p_n^{2d_n}$$

All the exponents $2d_i$ are EVEN.

10 points

- b) Let p be a prime. Prove, by contradiction, that the equation $px^2 = y^2$ does not have any positive integer solutions x, y . Hint: Use part 8a.

Suppose that x, y are positive integers and

$$px^2 = y^2.$$

The exponent e of p in the prime decomposition of y^2 is EVEN, by part a (possibly 0),

The exponent d of p in the prime decomposition of x^2 is even (possibly zero), hence the exponent of p in the prime decomposition of px^2 is $d+1$, which is ODD. Hence $\boxed{d+1 \neq e}$.

The equality $px^2 = y^2$ implies that $\boxed{d+1 = e}$, by the uniqueness of the prime decomposition.

A contradiction. ⁵

QED

c) (Bonus 5 points) Prove, by contradiction, that if p is prime then \sqrt{p} is not a rational number.

Suppose \sqrt{p} is a rational number.

Then there exist integers x, y , such that $\sqrt{p} = \frac{y}{x}$ (where $x \neq 0$).

Then $p = \frac{y^2}{x^2}$ and $p x^2 = y^2$ has positive integer solutions (we may replace x by $|x|$ and y by $|y|$). This contradicts part 8b. QED