Math 300 Section 2 Final Exam Spring 2016

Name:__________________________

Justify **all** your work!

1. (a) (5 points) Let the universe of discourse be the real numbers. Prove or disprove the following statement.

$$\exists x \; \forall y \; ((x+y)^2 + 5(x+y) + 6 > 0).$$

   (b) (5 points) Let the universe of discourse be the positive integers. Write the contrapositive of the following statement.
   If $p|ab$, then ($p|a$ OR $p|b$).

2. (a) (10 points) Let X be a set, $S \subset X$ a subset, and $S^c := \{x \in X \; : x \notin S\}$ the complementary subset. Assume that $\sharp(S) = \sharp(\mathbb{P})$ and $\sharp(S^c) = \sharp(\mathbb{P})$, i.e., the cardinalities of both S and S^c are equal to the cardinality of the set $\mathbb{P}$ of positive integers. Prove that $\sharp(X) = \sharp(\mathbb{P})$.
   Hint: By assumption, there exist bijections $f : S \to \mathbb{P}$ and $g : S^c \to \mathbb{P}$. Use them to construct a bijection $h : X \to \mathbb{P}$. **Prove** that your h is bijective.

   (b) (10 points) Regard the set $\mathbb{Q}$ of rational numbers as a subset of the real numbers $\mathbb{R}$ and let $\mathbb{Q}^c$ be the set of irrational real numbers. Prove that $\sharp(\mathbb{Q}^c) \neq \sharp(\mathbb{P})$ (so that $\mathbb{Q}^c$ is uncountable). Hint: Use part 2a.

3. (10 points) Use Mathematical Induction to prove that for all $n \geq 1$,

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \left(\frac{n+2}{2^n}\right).$$

4. (10 points) Set $\mathbb{P}_n = \{1, 2, \ldots, n\}$, where $n \geq 3$. Consider the permutations $\rho := \left(\begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{array}\right)$ and $\sigma := \left(\begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{array}\right)$. Note that ρ is a cyclic shift one position to the right, and σ reverses the order. Compute ρ^{-1} and $\sigma \circ \rho \circ \sigma$. Suggestion: Do it first for $n = 4$ so that $\rho := \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{array}\right)$ and $\sigma := \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{array}\right)$.

5. (10 points) Find the remainder when 5^{66183} is divided by 99. State any theorem you use and carefully justify your answer!

6. (10 points) Find the multiplicative inverse of 80 in $\mathbb{Z}_{253}$. Justify your answer. A complete answer will involve the Extended Euclidian Algorithm.

7. (10 points) Solve the simultaneous congruences

$$\begin{aligned} 3x &\equiv 10 \pmod{41} \qquad (1) \\ x &\equiv 20 \pmod{23} \qquad (2) \end{aligned}$$

8. (10 points) Consider the relation R on the set of real numbers defined by xRy if and only if $x - y$ is an integer. Prove that R is an equivalence relation.

9. (10 points) Prove that there are precisely three congruence classes in $\mathbb{Z}_{280}$, which solve the congruence

$$x^3 \equiv 13 \pmod{280}.$$

   You are not asked to find the solutions. State any theorem you use and carefully justify your answer.