

There are totally 9 problems in this exam.

Show all your work !!!

1. (15 pts) Let P, Q be statements. Show that the statement NOT (P OR Q) is equivalent to the statement (NOT P) AND (NOT Q) by showing they have the same truth tables.

P	Q	$\sim P$	$\sim Q$	$\sim(P \vee Q)$	$(\sim P) \wedge (\sim Q)$
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

the same

2. (10 pts) Prove or give a counterexample to the following statement:

$$\forall x \in \mathbb{Z}, \quad 2x^3 + 2x + 5 \neq 0.$$

Proof by contradiction. Suppose there is an integer solution $x_0 \in \mathbb{Z}$. Then $2x_0^3 + 2x_0 + 5 = 0$. This implies $2(x_0^3 + x_0) = -5$. which is contradiction because LHS is even but RHS = -5 which is odd.

3. (10 pts) Prove, using the contrapositive method, that

$$\text{if } 2x^3 + 3x^2 + 5x - 10 \leq 0, \text{ then } x \leq 1.$$

By contrapositive method, we assume $x > 1$.

$$\begin{aligned} \text{Then } 2x^3 + 3x^2 + 5x - 10 &> 2 \cdot 1^3 + 3 \cdot 1^2 + 5 \cdot 1 - 10 \\ &= 2 + 3 + 5 - 10 = 0 \end{aligned}$$

This is true because for $x > 1$, x^3, x^2, x are increasing.

Hence if $x > 1$, then $2x^3 + 3x^2 + 5x - 10 > 0$, which, by contrapositive method, shows that

if $2x^3 + 3x^2 + 5x - 10 \leq 0$, then $x \leq 1$.

4. (10 pts) Define the sequence x_n as follows:

$$x_1 = x_2 = 6, \text{ and for } n \geq 3, x_n = 2x_{n-1} + 3x_{n-2}.$$

Prove that for all $n \geq 1$, $x_n = 3(3^{n-1} + (-1)^{n-1})$.

Prove by Math Induction.

Step 1: Verify for $n=1, 2$. For $n=1$, LHS = $x_1 = 6$

$$\text{RHS} = 3 \cdot (3^{1-1} + (-1)^{1-1}) = 3 \cdot (1+1) = 6$$

$$\text{For } n=2, \text{ LHS} = x_2 = 6, \text{ RHS} = 3(3^{2-1} + (-1)^{2-1}) = 3 \cdot (3^1 + 1) = 6.$$

Step 2: Assume ~~$x_k = 3(3^{k-1} + (-1)^{k-1})$~~

$$x_k = 3(3^{k-1} + (-1)^{k-1}).$$

$$\begin{aligned} \text{Then } x_{k+1} &= 2x_k + 3x_{k-1} = 2 \cdot 3(3^{k-1} + (-1)^{k-1}) + 3 \cdot 3(3^{k-2} + (-1)^{k-2}) \\ &= 6(3^{k-1} + (-1)^{k-1}) + 9(3^{k-2} + (-1)^{k-2}) \\ &= 2 \cdot 3^k - 6 \cdot (-1)^k + 9 \cdot 3^k + 9 \cdot (-1)^k \\ &= 3 \cdot 3^k + 3 \cdot (-1)^k = 3(3^k + (-1)^k). \end{aligned}$$

5. (10 pts) Determine whether the following is true or not, and explain why.

$\text{NOT } (\forall x \in D, P(x) \Rightarrow Q(x))$ is equivalent to $\exists x \in D, (\text{NOT } P(x)) \text{ AND } Q(x)$.

They are not equivalent.

$$\text{Explanation: } \sim(\sim(\forall x \in D, P(x) \Rightarrow Q(x))) = \forall x \in D, P(x) \Rightarrow Q(x)$$

$$\begin{aligned} \sim(\exists x \in D, (\sim(P(x)) \wedge Q(x))) &= \forall x \in D, \sim(\sim P(x) \wedge Q(x)) \\ &= \forall x \in D, Q(x) \Rightarrow P(x) \end{aligned}$$

$$(\forall x \underset{\in D}{\cancel{\wedge}} P(x) \Rightarrow Q(x)) \neq (\forall x \in D, Q(x) \Rightarrow P(x))$$

6. (10 pts) Find an expression for

$$S_n = 1 - 2 + 3 - 4 + \dots + (-1)^{n+1} n, \text{ where } n = 1, 2, 3, \dots,$$

and prove the expression for S_n is correct.

Suppose $n = 2k$ is even, then

$$\begin{aligned} S_n = S_{2k} &= 1 - 2 + 3 - 4 + \dots + (-1)^{2k+1} 2k \\ &= (1-2) + (3-4) + \dots + ((2k-1) - (2k)) \\ &= (-1) + (-1) + \dots + (-1) = k \cdot (-1) = -k \end{aligned}$$

when $n = 2k+1$ odd, then

$$\begin{aligned} S_n = S_{2k+1} &= 1 - 2 + 3 - 4 + \dots + (-1)^{2k+1+1} (2k+1) \\ &= (1-2) + (3-4) + \dots + ((2k-1) - 2k) + (2k+1) \\ &= (-1) + (-1) + \dots + (-1) + (2k+1) = -k + 2k+1 \\ &= k+1. \end{aligned}$$

$$\text{so } S_n = \begin{cases} \frac{-n}{2} & \text{if } n = \text{even} \\ \frac{n+1}{2} & \text{if } n = \text{odd} \end{cases}$$

7. (15 pts) Consider the linear Diophantine equation: $10x + 25y = 200$.

(1) Find all integer solutions of the equation.

(2) Find all non-negative integer solutions of the equation.

Step 1: Solve $10x + 25y = 5$. By guessing, $\tilde{x} = -2$, $\tilde{y} = 1$ is a solution. From this we obtain one particular solution (x_0, y_0) of $10x + 25y = 200$, where

$$\begin{cases} x_0 = \tilde{x} \cdot \frac{200}{5} = (-2) \cdot \frac{200}{5} = -80 \\ y_0 = \tilde{y} \cdot \frac{200}{5} = 1 \cdot \frac{200}{5} = 40 \end{cases}$$

Step 2: Find all solutions:

$$\begin{cases} x = x_0 + \frac{-b}{\gcd} m = -80 + \frac{-25}{5} m = -80 - 5m \\ y = y_0 + \frac{a}{\gcd} m = 40 + \frac{10}{5} m = 40 + 2m, \quad m \in \mathbb{Z} \end{cases}$$

Step 3: Find all non-negative solutions:

$$\begin{cases} x \geq 0 \\ y \geq 0 \end{cases} \Rightarrow \begin{array}{l} -80 - 5m \geq 0 \\ 40 + 2m \geq 0 \end{array} \Rightarrow -20 \leq m \leq -16$$

so $m = -16, -17, -18, -19, -20$, and

$$\boxed{\begin{array}{l} \cancel{\begin{cases} x = 0, 5, \\ y = 8, 6 \end{cases}} \\ (x, y) = (0, 8), (5, 6), (10, 4), \\ (15, 2), (20, 0) \end{array}}$$

8. (10 pts) Determine whether 223 is prime or not, and explain why.

$\sqrt{223} < 15$. So we check whether p is a divisor of 223 for $p = 2, 3, 5, 7, 11, 13$.
 $2 \nmid 223, 3 \nmid 223, 5 \nmid 223, 7 \nmid 223,$
 $11 \nmid 223, 13 \nmid 223$. Hence 223 is prime.

9. (10 pts) Let $u, v \in \mathbb{Z}$. Suppose $\gcd(3, u) = 1$. Prove that if $u \mid 9v$, then $u \mid v$.

Since $q = 3^2$, $\gcd(3, u) = 1 \Rightarrow \gcd(q, u) = 1$.

Now there are $x, y \in \mathbb{Z}$, such that

$$qx + uy = 1.$$

Then $v = v \cdot 1 = v \cdot (qx + uy) = x \cdot (qv) + (vy) \cdot u$
which means that v is a linear combination
of qv and u . Since $u \mid qv$ and $u \mid u$,
we have $u \mid v$.