## Math 300 - Exam 1 Solutions <br> March 4, 2014

1. Give precise and complete definitions of the following:
(a) (3 points) The intersection the sets $S$ and $T$

Solution: $S \cap T=\{x \mid(x \in S) \wedge(x \in T)\}$.
The intersection of $S$ and $T$ is the set consisting of the elements that belong to both $S$ and $T$.
(b) (3 points) A statement or proposition

Solution: is a declarative sentence that is unambiguously true or false.
(c) (3 points) The binomial coefficient $\binom{n}{r}$

Solution: is $\frac{n!}{r!(n-r)!}$. (We showed in class that $\binom{n}{r}$ is the number of subsets of size $r$ contained in a given set of size $n$. This can also be used as the definition of $\binom{n}{r}$, though it's not the definition given in the text.)
(d) (3 points) A statement $P$ implies a statement $Q$

Solution: if and only if $Q$ is true whenever $P$ is true. ( $Q$ may also be true when $P$ is false.) In terms of the logical operators, $P \Longrightarrow Q$ is equivalent to $(\neg P) \vee Q$.
2. (3 points) Give a precise and complete statement of the Principle of Strong Induction.

Solution: Let $P(n)$ be a statement that depends on the natural number $n$. If $P$ satisfies the conditions
(i) $P(1)$ is true, and
(ii) for all $k \in \mathbb{N}$, if $P(j)$ is true for every $j$ satisfying $1 \leq j \leq k$, then $P(k+1)$ is true, then $P(n)$ is true for all $n \in N$.
This problem was on the quiz given on February 20.
3. (7 points) Write the truth table for the statement $(P \wedge Q) \Longrightarrow \neg R$. (You don't have to show any work except the truth table.)

|  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $P$ | $Q$ | $R$ | $P \wedge Q$ | $\neg R$ | $(P \wedge Q) \Longrightarrow \neg R$ |  |
|  | T | T | T | T | F | F |
|  | T | T | F | T | T | T |
|  | T | F | T | F | F | T |
|  | T | F | F | F | T | T |
|  | F | T | T | F | F | T |
| F | T | F | F | T | T |  |
| F | F | T | F | F | T |  |
| F | F | F | F | T | T |  |

4. (18 points) ( 2 points for each correct answer, -1 point for each wrong answer.)

Determine whether the following statements are true ( $\mathbf{T}$ ) or false ( $\mathbf{F}$ ). Circle the correct answer. You don't have to justify your choice.

For parts (a)-(e), let $A\{1,\{2,3\},\{4\}\}$.
(a) $2 \in A$
(b) $\emptyset \in A$
(c) $\{1,\{4\}\} \subseteq A$
T
T
(T) $\mathbf{F}$
(d) $\{1\} \subseteq A$
(T) $\mathbf{F}$
(e) $\emptyset \nsubseteq A$
(f) The negation of the statement "Every even integer greater than two can be written as the sum of two prime numbers," is "There is an odd number which can be written as the sum of two prime numbers.
(g) If $A, B$, and $C$ are sets, then $(A \cup B) \cap C \subseteq A \cup(B \cap C)$
(h) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x+y=0$
5. (10 points) We discussed the universal and existential quantifiers in class. Often, we want to say that there is exactly one element $x$ in a set $X$ for which the predicate $P(x)$ true. For this, we have the unique existential quantifier $\exists$ ! and we write $\exists!x \in X, P(x)$. One way to define it in terms of the ordinary existential and universal quantifiers is:

$$
\exists!x \in X, P(x) \quad \Longleftrightarrow \quad \exists x \in X,(P(x) \wedge(\forall y \in X,(P(y) \Longrightarrow x=y)))
$$

Use this to write a logical expression for $\neg \exists!x \in X, P(x)$. Simplify your expression as much as you can.

## Solution:

$$
\begin{aligned}
\neg(\exists!x \in X, P(x)) & \equiv \neg(\exists x \in X,(P(x) \wedge(\forall y \in X,(P(y) \Longrightarrow x=y)))) \\
& \equiv \forall x \in X, \neg(P(x) \wedge(\forall y \in X,(P(y) \Longrightarrow x=y))) \\
& \equiv \forall x \in X, \neg P(x) \vee \neg(\forall y \in X,(P(y) \Longrightarrow x=y)) \\
& \equiv \forall x \in X, \neg P(x) \vee \exists y \in X, \neg(P(y) \Longrightarrow x=y) \\
& \equiv \forall x \in X, \neg P(x) \vee \exists y \in X,(P(y) \wedge x \neq y)
\end{aligned}
$$

This says that, for all $x \in X$, either $P(x)$ is false or there's some other element $y$ with $P(y)$ true. So one possibility is that $P(x)$ is false for all $x$. Otherwise, there's at least one $x$ for which $P(x)$ is true, but there's a $y$ not equal to that $x$ for which $P(y)$ is also true.
6. Let $P$ be the statement "Today is Tuesday"; $Q$ be the statement "I am in Italy"; $R$ be the statement "I will go to a great restaurant"; and $S$ be the statement "It is a sunny day".
(a) (5 points) Give a logical formula equivalent to the statement "If I am in Italy it is a sunny day and I will go to a great restaurant."

Solution: $Q \Longrightarrow(S \wedge R)$. (Some people read the statement as "If I am in Italy, it is a sunny day. And I will go to a great restaurant." So they got $(Q \Longrightarrow S) \wedge R$. I accepted that, but I don't think that's how most native speakers of English, at least, would interpret the given sentence.
(b) (5 points) Express this statement in English: $\neg Q \Longrightarrow \neg R$.

Solution: If I am not in Italy, then I will not go to a great restaurant.
7. (15 points) Use induction to prove that, for every natural number $n$,

$$
\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right) \cdots\left(1+\frac{1}{n}\right)=n+1
$$

Solution: Let $P(n)$ be the statement that

$$
\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)=n+1
$$

We will use induction to prove that this statement holds for all natural numbers $n$.

The base case is $P(1)$. We have

$$
\prod_{i=1}^{1}\left(1+\frac{1}{i}\right)=1+\frac{1}{1}=2
$$

We also have $n+1=2$. So $P(1)$ is true.
For the induction step, we must prove that, for any natural number $k, P(k) \Longrightarrow P(k+1)$. So we assume that $P(k)$ is true and use that to try to prove $P(k+1)$ is true (without using anything about $k$ except that it's a natural number). Thus, we start from the assumption that

$$
\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)=k+1
$$

and we want to prove that

$$
\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)=(k+1)+1
$$

So consider

$$
\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)=\left(\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)\right)\left(1+\frac{1}{k+1}\right)
$$

By the assumption about $P(k)$, we know the right-hand side is

$$
\begin{aligned}
& =(k+1)\left(1+\frac{1}{k+1}\right) \\
& =(k+1)+\frac{k+1}{k+1} \\
& =k+1+1 \\
& =k+2 .
\end{aligned}
$$

So we have proved that $P(k) \Longrightarrow P(k+1)$ for all natural numbers $k$. By the Principle of Mathematical Induction, this tells us that $P(n)$ is true for all natural numbers $n$.
8. (15 points) Prove that $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} 2^{2 k}=3^{n}$. (Hint: Use the Binomial Theorem.)

Solution: Taking the hint, we try to apply the Binomial Theorem. Examining the expression on the left, we see that it is a sum of terms involving $(-1)^{n-k}$ and $2^{2 k}$, along with the binomial coefficient $\binom{n}{k}$. If the power of 2 in each term were $k$, instead of $2 k$, this would just be the binomial expansion of $(-1+2)^{n}=1^{n}$. But $2^{2 k}=\left(2^{2}\right)^{k}$, so we do have the expansion of $\left(-1+2^{2}\right)^{n}$. And, of course, $-1+2^{2}=3$. We can state this efficiently as follows.
Note that $3=-1+4$, so $3^{n}=(-1+4)^{n}$. Applying the binomial theorem to the last expression, we get

$$
3^{n}=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} 4^{k}
$$

Since $4^{k}=2^{2 k}$, we have proved the desired result.
9. (10 points) Let $n$ be a natural number. Prove that if $n^{2}$ is divided by 4 , the remainder is either 0 or 1 .

Solution: Every natural number is either even or odd. We consider these cases separately.
Suppose $n$ is even, so $n=2 k$ for some natural number $k$. Then $n^{2}=4 k^{2}$. Clearly, if we divide $4 k^{2}$ by 4 , the remainder is 0 .
If $n$ is not even, it must be odd. This means that $n=2 m-1$ for some natural number $m$. Then $n^{2}=(2 m-1)^{2}=4 m^{2}-4 m+1=4\left(m^{2}-m\right)+1$. Since $n^{2}$ is 1 more than a multiple of 4 , dividing $n^{2}$ by 4 leaves a remainder of 1 .
This completes the proof.

