

Two additional examples with quantifiers

Ex: Let the universe of discourse be  $\mathbb{R}$  (real numbers)

Q. Is the following statement true or false:

$$\neg \exists M \forall x (x^2 \leq M)''$$

A. The statement is false

Let us prove that

\*  $\text{Not}(\exists M \forall x x^2 \leq M)$   
is true.

\*  $\Leftrightarrow \forall M \text{ Not}(\forall x x^2 \leq M)$   
 $\Leftrightarrow \forall M \exists x (x^2 > M).$

It remains to prove that

$\forall M \exists x, x^2 > M$   
is true.

Given  $M$ , take  $x = \sqrt{|M|+1}$ .

Then  $x^2 = |M| + 1 > M$ .

Q.E.D

$$\boxed{\begin{aligned} \text{Not}(\exists M \forall x, P(x)) &\Leftrightarrow \\ \forall M \exists x, \text{Not}(P(x)). \end{aligned}}$$

Ex: Determine whether the following statements are always equivalent:

(a)  $\forall x (P(x) \text{ OR } Q(x))$

(b)  $(\forall x, P(x)) \text{ OR } (\forall x, Q(x))$ .

The statements are NOT equivalent.  
Example 1: Take the universe of discourse to  $\mathbb{Z}$  (integers). Let

$P(x)$  mean " $x$  is even"

$Q(x)$  mean " $x$  is odd".

Statement (a)  $\forall x \in \mathbb{Z} (x \text{ is even OR } x \text{ is odd})$

(b)  $(\forall x \in \mathbb{Z}, x \text{ is even}) \text{ OR } (\forall x \in \mathbb{Z}, x \text{ is odd})$

In this case statement (a) is true, but statement (b) is False.

Example 2: Let the universe of discourse be  $\mathbb{R}$ ,

Let  $P(x)$  be " $x > 0$ "

Let  $Q(x)$  be " $x \leq 0$ ".

(a)  $\forall x \in \mathbb{R}, (x > 0 \text{ OR } x \leq 0)$

(b)  $(\forall x \in \mathbb{R}, x > 0) \text{ OR } (\forall x \in \mathbb{R}, x \leq 0)$

## Section 1.5 Proofs

standard proof techniques.

Direct proof method: We prove  
 $P \Rightarrow Q$ .

Proposition: Let  $S, T$  be sets.

If  $S \cup T = S$ , then  $T \subseteq S$ .

Proof: Assume that  $S \cup T = S$ .

Let  $x \in T$ . Then  $x \in \underbrace{S \cup T}_{\text{"}} \subseteq S$ . Now

$\{y : y \in S \text{ or } y \in T\}$

$S \cup T = S$ , so  $x \in S$ .

We showed that  $x \in T \Rightarrow x \in S$ .

This means that  $T \subseteq S$ .

QED

Proving  $P \Leftrightarrow Q$ .

This requires proving that

$P \Rightarrow Q$  and  $Q \Rightarrow P$ .

Example:

Proposition: Let  $S, T$  be sets,  
 $S \cap T = \emptyset$  if and only if  $S \subseteq T$ .

P

Q

Proof: ( $\Rightarrow$ ) Assume that

$S \cap T = \emptyset$ . Let  $x \in S$ . Then

$x \in S \cap T$ , because  $S = S \cap T$  is

assumed. So  $x \in \emptyset$  AND  $x \in T$  is  
true.  
So  $x \in T$ . We conclude that

$S \subseteq T$ .

( $\Leftarrow$ ) Assume that  $S \subseteq T$ .

We need to show

(a)  $S \cap T \subseteq \emptyset$ .

(b)  $S \subseteq S \cap T$ .

Statement (a) is true for any pair of  
sets. It remains to prove (b).

Let  $x \in S$ . We assumed that  $S \subseteq T$ ,  
 So  $x \in T$  as well, So  $x \in S \cap T$ ,  
 We conclude - that (b) is true,

Q.E.D

The **CONTRAPOSITIVE PROOF METHOD:**

The Contrapositive Law: The statement

a)  $P \Rightarrow Q$

b)  $\text{Not } Q \Rightarrow \text{Not } P$ .

are equivalent.

P	Q	$P \Rightarrow Q$	$\text{Not } Q$	$\text{Not } P$	$(\text{Not } Q) \Rightarrow (\text{Not } P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

SAME

Ex: Use the contrapositive method  
to prove: Let  $R, S, T$  be sets,

$$R \cap S = \emptyset \Rightarrow ((R \cup T) \cap (S \cup T)) \subset T.$$

Proof:

We will prove the contrapositive

$$((R \cup T) \cap (S \cup T)) \not\subset T \Rightarrow R \cap S \neq \emptyset.$$

Assume  $((R \cup T) \cap (S \cup T)) \not\subset T$ . Then

there exists  $x \in (R \cup T) \cap (S \cup T)$ ,

such that  $x \notin T$ . So

$$(x \in R \cup T) \text{ AND } (x \in S \cup T) \text{ AND } (x \notin T)$$
$$\Leftrightarrow$$

$$(x \in R \text{ OR } x \in T) \text{ AND } (x \in S \text{ OR } x \in T)$$
$$\text{AND } (x \notin T).$$

$$\text{So } x \in R \text{ AND } x \in S.$$

Hence  $x \notin R \cap S$ .

So  $R \cap S \neq \emptyset$ . Q.E.D

## PROOF BY CONTRADICTION:

We assume that the statement needed to be proven is false and use this assumption to arrive at a contradiction.

Example: We will use the fact from middle school:

Def: An integer  $p > 1$  is PRIME, if the only positive integers dividing  $p$  are 1 and  $p$ .

(4) Fact: If  $n$  is an integer  $\geq 1$ , then there exists a prime  $p \geq 2$  such that  $p$  divides  $n$ .

Theorem: There are infinitely many primes.

Proof: (by contradiction)

Assume that there are only finitely many positive primes. So we can list them all

$$P_1, P_2, \dots, P_K.$$

Let  $g = 1 + \underbrace{P_1 \cdot P_2 \cdots P_K}_{\text{the product of all the prime.}}$ .

Then  $g$  is an integer  $\geq 2$ .

By (†) we know that there exists a prime  $P \geq 2$ , such that  $P$  divides  $g$ . But  $P = P_i$ , for some  $1 \leq i \leq K$ , by our assumption, so  $P$  divides  $g - 1$ . So

$$P \text{ divides } g - (g - 1) = 1.$$

This contradicts the fact that  $P \geq 2$ . Hence, the statement is proven, by the method of proof by contradiction. QED

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Proving " $P \Rightarrow (Q \text{ OR } R)$ :

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The above statement is equivalent to

" $\neg(P \text{ AND } \neg Q) \Rightarrow R$ ".

In the homework you will check that their truth tables are the same.

Proposition: Let  $a$  and  $b$  be integers.

If  $a^3 + b$  is odd,

P

then Q  
a is odd OR  
b is odd.

R

Proof: The statement is equivalent to  
 $(a^3 + b \text{ is odd}) \text{ AND } (a \text{ is even}) \Rightarrow$

Assume that  $a^3 + b$  is odd and  $a$  is even. Then  $a^3$  is even as well. So the parity of  $a^3 + b$  is the same as the parity of  $b$ . So  $b$  is odd as well.

QED

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Proof by counter example:

Consider a statement of the form " $\forall x, P(x)$ ".

In order to show that the above is false, it suffices to find one  $x$ , such that  $P(x)$  is false.

I.e.  $\forall x, P(x)$

$\Leftrightarrow \exists x, (\text{Not } P(x))$

Example: Disprove:

$\Rightarrow$  Every positive integer is  
the sum of three squares  
(of integers)

Proof: It suffices to come up  
with one positive integer, which  
is not the sum of 3 squares.

Consider 7.

Assume <sup>that</sup>  $a, b, c$  are non-negative  
integers and  $a^2 + b^2 + c^2 < 8$ .

[We will show that  $a^2 + b^2 + c^2 \leq 6$ .]

Then  $a, b, c \leq 2$  and at  
most one of them is 2.

So we may assume that

$a \in \{0, 1\}$ ,  $b \in \{0, 1\}$ ,  $c \in \{0, 1, 2\}$ ,

Then  $a^2 + b^2 + c^2 \leq 1^2 + 1^2 + 2^2 = 6$ .

$\begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \quad \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \quad \begin{matrix} 1 \\ 1 \\ 2 \end{matrix}$

QED