

MATH 300 MIDTERM FALL 2015

There are totally 9 problems in this exam.

Show all your work !!!

1. (15 pts) Let P, Q be statements. Show that the statement NOT (P OR Q) is equivalent to the statement (NOT P) AND (NOT Q) by showing they have the same truth tables.

2. (10 pts) Prove or give a counterexample to the following statement:

$$\forall x \in \mathbb{Z}, \quad 2x^3 + 2x + 5 \neq 0.$$

3. (10 pts) Prove, using the contrapositive method, that

$$\text{if } 2x^3 + 3x^2 + 5x - 10 \leq 0, \text{ then } x \leq 1.$$

4. (10 pts) Define the sequence x_n as follows:

$$x_1 = x_2 = 6, \text{ and for } n \geq 3, \quad x_n = 2x_{n-1} + 3x_{n-2}.$$

Prove that for all $n \geq 1$, $x_n = 3(3^{n-1} + (-1)^{n-1})$.

5. (10 pts) Determine whether the following is true or not, and explain why.

$\text{NOT } (\forall x \in D, P(x) \Rightarrow Q(x))$ is equivalent to $\exists x \in D, ((\text{NOT } P(x)) \text{ AND } Q(x))$.

6. (10 pts) Find an expression for

$$S_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n+1}n, \text{ where } n = 1, 2, 3, \dots,$$

and prove the expression for S_n is correct.

7. (15 pts) Consider the linear Diophantine equation: $10x + 25y = 200$.

(1) Find all integer solutions of the equation.

(2) Find all non-negative integer solutions of the equation.

8. (10 pts) Determine whether 223 is prime or not, and explain why.

9. (10 pts) Let $u, v \in \mathbb{Z}$. Suppose $\gcd(3, u) = 1$. Prove that if $u \mid 9v$, then $u \mid v$.

$$1) ((\text{Not } P) \text{ AND } (\text{NOT } Q))$$

\Leftrightarrow $\text{Not } (P \text{ OR } Q)$. $(\text{NOT } Q)$

| P | Q | P OR Q | NOT(P OR Q) | NOT P | NOT Q | (NOT P) AND (NOT Q) |
|---|---|--------|-------------|-------|-------|------------------------|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

↑ ↑

SAME, so statements

(2) ~~Direct Proof~~ are equivalent.
method.)

$$\forall x \in \mathbb{Z}, 2x^3 + 2x + 5 \neq 0.$$

The statement is true.

If x is an integer, then

$2x^3 + 2x$ is even, so

$2x^3 + 2x + 5$ is odd, so not

equal to 0, because 0 is even.

(3) Prove, using the contrapositive method, that $\text{if } 2x^3 + 3x^2 + 5x - 10 \leq 0, \text{ then } x \leq 1$.

The contrapositive statement of " $P \Rightarrow Q$ " is the equivalent statement " $\text{Not } Q \Rightarrow \text{Not } P$ ".

Contrapositive: If $x > 1$, then $2x^3 + 3x^2 + 5x - 10 > 0$.

Equivalently,

\therefore If $x > 1$, then $2x^3 + 3x^2 + 5x > 10$.

Indeed, if $x > 1$, then

$$2x^3 > 2, \quad 3x^2 > 3, \quad 5x > 5, \quad \text{so}$$

$$2x^3 + 3x^2 + 5x > 2 + 3 + 5 = 10.$$

(4) Recursive definition of $\{x_n\}$.

$x_1 = x_2 = 6$, and for $n \geq 3$, $x_n = 2x_{n-1} + 3x_{n-2}$.

Prove that for all $n \geq 1$, $x_n = 3(3^{n-1} + (-1)^{n-1})$.

Proof by Strong Induction:

Initial cases: $n=1: x_1 = 6 = 3(3^0 + (-1)^0)$. True.

$$\underline{n=2:} x_2 = 6 = 3(3^1 + (-1)^1)$$

True.

Induction Step: Let $n \geq 2$.

Strong Induction Hypothesis:

Assume that the statement $P(k)$ is " $x_k = 3(3^{k-1} + (-1)^{k-1})$ "

is true for k positive integers $\leq n$.

We need to prove $P(n+1)$. I.e., the equality: $x_{n+1} = 3(3^n + (-1)^n)$.

$$x_{n+1} = 2x_n + 3x_{n-1} =$$

We are given that
 $x_n = 2x_{n-1} + 3x_{n-2}$.

\uparrow
 Induction
 Hypothesis

$$\begin{aligned}
 & 2\left(3\left(3^{n-2} + (-1)^{n-2}\right)\right) + 3\left(3\left(3^{n-2} + (-1)^{n-2}\right)\right) = \\
 & \quad \underbrace{x_n}_{\text{using } P(n)} \quad \underbrace{x_{n-1}}_{\text{using } P(n-1)} \\
 & = \underbrace{2 \cdot 3^n + 3^n}_{(2+1)3^n} + \underbrace{(-6+9)}_3 \underbrace{(-1)^{n-2}}_{(-1)^n} = \\
 & \quad \underbrace{3^{n+1}}_{\text{ }} \\
 & = 3(3^n + (-1)^n). \quad \text{Q.E.D}
 \end{aligned}$$

(5) True or False?

NOT $(\forall x \in D, P(x) \Rightarrow Q(x))$ is equivalent to $\exists x \in D, ((\text{NOT } P(x)) \text{ AND } Q(x))$.

Recall: "Not $(\forall x, R(x))$ "

\Leftrightarrow " $\exists x, \text{Not } R(x)$ "

So the LHS $\textcircled{*}$ is equivalent to
" $\exists x \in D, \text{Not } (P(x) \Rightarrow Q(x))$ "

\Downarrow
 $P(x) \text{ AND } (\text{NOT } Q(x))$

So $\textcircled{*}$ is equivalent to

" $\exists x \in D, P(x) \text{ AND } (\text{NOT } Q(x))$ ".

This is different from $\textcircled{**}$.

Our guess is that $\textcircled{*}$ and $\textcircled{**}$
are NOT equivalent, look for
a counterexample.

NOT $(\forall x \in D, P(x) \Rightarrow Q(x))$ is equivalent to $\exists x \in D, ((\text{NOT } P(x)) \text{ AND } Q(x))$.

Counter Example 1: Let $D = \mathbb{R}$ Real numbers.

Let $P(x)$ be $x > 1$,

Let $Q(x)$ be $x > 0$.

So $\textcircled{1}$ is False.

Now $\textcircled{2}$ states

" $\exists x \in \mathbb{R}, x \leq 1 \text{ AND } x > 0$ ".

This is true, take $x = \frac{1}{2}$.

So $\textcircled{2}$ is true.

So $\textcircled{1}$ and $\textcircled{2}$ are NOT equivalent.

Counter example 2: Take $D = \mathbb{Z}$
 $P(x)$ is " $4|x$ " and $Q(x)$ is " $2|x$ ".

Then ① is false and ② is true.

(6)

$$S_n = 1 - 2 + 3 - 4 + \dots + (-1)^{n+1}n, \text{ where } n = 1, 2, 3, \dots,$$

the expression for S_n is correct.

| | | | | | | | |
|-------|---|---|---|----|---|----|---------|
| n | 1 | 2 | 3 | 4 | 5 | 6 | |
| S_n | 1 | $1-2$ $\underbrace{\quad\quad}_{-1}$ | $1-2+3$ $\underbrace{\quad\quad\quad}_{2}$ | -2 | 3 | -3 | \dots |

If n is even $n=2k$, then $S_{2k} = -k$

If n is odd $n=2k-1$, $k \geq 1$, then

$$S_{2k-1} = k.$$

$$S_n = \begin{cases} -k & \text{if } n = 2k \text{ (even)} (k \geq 1) \\ k & \text{if } n = 2k-1 \text{ (odd)} (k \geq 1), \end{cases}$$

Note: $S_1 = 1$ and

$$S_n = S_{n-1} + (-1)^{n+1} n.$$

A recursive definition for S_n .

We will prove by induction each of the two statements separately.

Statement A₀^(K) $S_{2K} = -K$, $K \in \mathbb{P}$.

Statement B₀^(K) $S_{2K-1} = K$, $K \in \mathbb{P}$.

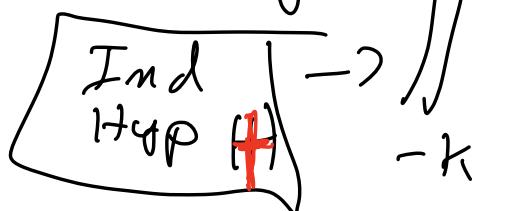
Proof of Statement A by induction,

Case $K=1$

$$\frac{S_2 = -1}{\text{above table}} \text{ so } A(1) \text{ is true.}$$

Assume $A(K)$,

$$S_{2(K+1)} = S_{2K} + (-1)^{(2K+1)+1} \underbrace{(-1)^{(2K+1)} + (-1)^{(2K+2)}}_{(2K+1) - (2K+2)}^{(2K+1)+1}$$



(+)

$$\boxed{S_{2K} = -K}$$

$$\underbrace{(-1)^{(2K+1)} + (-1)^{(2K+2)}}_{(2K+1) - (2K+2)}$$

$$\underbrace{-1}_{-1}$$

$= -K - 1$. So $A(K+1)$ is true.

Exercise: Prove statement B
by induction similarly. QED

7. (15 pts) Consider the linear Diophantine equation: $10x + 25y = 200$.

(1) Find all integer solutions of the equation.

(2) Find all non-negative integer solutions of the equation.

(1) (i) First solve $\begin{matrix} 10 & x \\ \downarrow b & \downarrow a \end{matrix} + 25y = \gcd(10, 25)$.

Here we can easily guess a particular sol'n: $(y=2, x=-5)$.
In general, we need the EEA and we proceed as follows:

$$25y_i + 10x_i = r_i$$

| | y_i | x_i | r_i | q_i |
|---|-------|-------|-------|-------|
| 1 | 1 | 0 | 25 | — |
| 2 | 0 | 1 | 10 | — |
| 3 | 1 | -2 | 5 | 2 |

$\gcd(25, 10)$
is that

$$\begin{array}{ccccccc}
 & | & | & | & | & | & \\
 & & & 0 & & 2 & \\
 & | & | & \equiv & | & | & \\
 \frac{y_3}{m} & & & & \frac{x_3}{m} & & \\
 25 \cdot 1 + 10 \cdot (-2) = 5 & & & & & &
 \end{array}
 \quad \text{mod } -2020$$

$(x, y) = (-2, 1)$ is a particular
sol'n to

$$10x + 25y = 5 = \gcd(10, 25)$$

(ii) Solve $10x + 25y \stackrel{*}{=} 200$

A solution exists since

$\gcd(10, 25)$ divides 200.

A particular solution is $(x_0, y_0) = \frac{200}{5} \cdot (-2, 1) = (-80, 40)$.

In this case, all integer
solutions of $\stackrel{*}{=}$ are given by

$$(x, y) = \left(x_0 - \left(\frac{25}{5}\right) \cdot k, y_0 + \left(\frac{10}{5}\right) k \right), \quad k \in \mathbb{Z}$$

by a theorem in the textbook.

$$= (-80 - 5k, 40 + 2k), k \in \mathbb{Z}.$$

(2) All non-negative integers
Sol'n's: Need k to satisfy

$$(a) -80 - 5k \geq 0, \text{ and}$$

$$(b) 40 + 2k \geq 0,$$

$$(a) -80 \geq 5k \\ -16 \geq k$$

$$(b) 2k \geq -40 \\ k \geq -20.$$

$$\underbrace{(a) + (b)}_0; -20 \leq k \leq -16 \\ k = -16, -17, -18, -19, -20.$$

$$\text{If } k = -16, (x, y) = \left(\underbrace{-80 - 5(-16)}_0, \underbrace{40 + 2(-16)}_8 \right).$$

$$\text{If } k = -17, (x, y) = (5, 6)$$

$$k = -18, (x, y) = (19, 1)$$

$$k = -19, (x, y) = (13, 2)$$

$$k = -20, (x, y) = (20, 0)$$

(8) Is 223 prime?

If 223 is NOT a prime, then there would be a prime $\leq \sqrt{223}$

which divides it.

Even numbers do not divide 223

It suffices to check if one of

$3, 5, 7, 11, 13$ divide 223

Use a calculator to check that

none divides, so 223 is a prime.

(9)

Let $u, v \in \mathbb{Z}$. Suppose $\gcd(3, u) = 1$. Prove that if $u \mid 9v$, then $u \mid v$.

Method 1:

Proof using the following Proposition proven in class.

Prop: Let a, b, c be integers.

If $c \mid ab$, and $\gcd(c, a) = 1$,
then $c \mid b$.

If $\gcd(3, u) = 1$, then

$\gcd(9, u) = 1$. Indeed, if

a prime p divides 9 and u ,

then p divides also 3 and u .

But there is no such a common division of 3 and u because

$\gcd(3, u) = 1$, By assumption.

So if $u \mid 9v$, then $u \mid v$, by

the proposition (with $c=u$,

$$\begin{array}{l} a=g \\ b=r \end{array}$$

Method 2:

The proof given on the website does not use the above prop but instead use its proof.

The key is that $\gcd(g, u) = 1$

implies (by the E.E.A Theorem)

that there exist integers

x, y , such that $gx + uy = 1$.

So

$$v = \sqrt{1} = \sqrt{(gx + uy)} =$$

$$= \underbrace{\sqrt{g}}_{\text{u}} x + \underbrace{\sqrt{u}}_{\text{u}} y$$

Now $u \mid gv$ (given) so u divides the first summand and u clearly

divides also the second summand

so u divides v .