

Name: My Solution

1. (15 points) Let P and Q be statements. Find the truth tables of the following statements and use them to determine if they are equivalent.

i) $P \Rightarrow Q$.

ii) $(\text{NOT } P) \text{ OR } (\text{NOT } Q) \Rightarrow (\text{NOT } P)$.

P	Q	$P \Rightarrow Q$	$\text{NOT } P$	$\text{NOT } Q$	$(\text{NOT } P) \text{ OR } (\text{NOT } Q)$	$(\text{NOT } P) \text{ OR } (\text{NOT } Q) \Rightarrow (\text{NOT } P)$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	T
F	F	T	T	T	T	T

SAME, so (i) \Leftrightarrow (ii)

2. (10 points) Let the universe of discourse be the real numbers. Prove the following statement: $\forall \epsilon > 0 \exists \delta > 0, |x - 2| < \delta \Rightarrow |3x - 6| < \epsilon$.

Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{3}$. Then

$$|x - 2| < \delta \Leftrightarrow |x - 2| < \frac{\epsilon}{3} \Rightarrow 3|x - 2| < \epsilon \Leftrightarrow |3x - 6| < \epsilon,$$

Hence for every $\epsilon > 0$, there exist $\delta > 0$ (namely $\delta = \frac{\epsilon}{3}$) such that the implication holds.

3. (10 points) Let the universe of discourse be the real numbers. Write first the contrapositive and then the converse of the following statement.
If $x^2 + y^2 = 25$, then $|x| \leq 5$.

contra positive: If $|x| > 5$, then $x^2 + y^2 \neq 25$.

Converse: If $|x| \leq 5$, then $x^2 + y^2 = 25$.

4. (15 points) Let S and T be sets. Prove or give a counter example.

$$S \cup T = T \Leftrightarrow (S \subset T). \quad \text{→ 3 pt}$$

The statement is correct. Def: $S \cup T = \{x : x \in S \text{ or } x \in T\}$

(\Rightarrow) Assume that $S \cup T = T$.

Let $x \in S$. Then $x \in S \cup T$. But $S \cup T = T$.

Hence $x \in T$, so $S \subset T$.

(\Leftarrow) Assume that $S \subset T$.

Let $x \in S \cup T$. Then $x \in S$ or $x \in T$.

If $x \in S$, then $x \in T$, since $S \subset T$. Hence
 $x \in T$ either way. So $\boxed{S \cup T \subset T}$

Let $t \in T$. Then $t \in S \cup T$, by def of the
union. Hence $\boxed{T \subset S \cup T}$. Thus $S \cup T = T$.

QED

5. (10 points) How many positive common divisors do 600 and 4500 have? Justify your answer!

The set of common divisors of a and b is precisely the set of divisors of $\gcd(a, b)$.

$$4500 = \underbrace{7 \cdot 600}_{4200} + 300 \quad \text{and } 300 \mid 600,$$

$$\text{So, } \gcd(600, 4500) = 300.$$

Prime decomposition:

$$300 = 3^1 \cdot 2^2 \cdot 5^2$$

If $c \mid 300$, then $c = 3^i \cdot 2^j \cdot 5^k$ where
 $0 \leq i \leq 1$, $0 \leq j \leq 2$, $0 \leq k \leq 2$.

There are $2 \cdot 3 \cdot 3 = 18$ possibilities for (i, j, k) .

Hence, there are 18 common divisors.

6. (10 points) Prove that $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$ if and only if $\gcd(ab, c) = 1$.

(\Leftarrow) Assume that $\gcd(ab, c) = 1$. If $d \mid a$, then $a = dq$, for some $q \in \mathbb{Z}$, $ab = d(bq)$, so $d \mid ab$. Hence, if d is a common divisor of a and c , then d is a common divisor of ab and c . So $d \leq \gcd(a, c) = 1$. Hence, $\gcd(a, c) = 1$.

Similarly, interchanging the roles of a and b , we get that $\gcd(b, c) = 1$.

(\Rightarrow) Assume that $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$.
Method 1: Let d be a common divisor of ab and c . Now $[\gcd(a, c) = 1 \text{ and } d \mid c] \Rightarrow \gcd(a, d) = 1$. Furthermore $\gcd(a, d) = 1$ and $d \mid ab \Rightarrow d \mid b$, by Proposition 2.28. Hence, d is a common divisor of b and c . Hence $d \leq \gcd(b, c) = 1$. So, $\gcd(ab, c) = 1$.

Method 2: The assumption implies that there exist integers x_1, y_1 such that $ax_1 + cy_1 = 1$ and integers x_2, y_2 such that $bx_2 + cy_2 = 1$. Hence, $1 = (ax_1 + cy_1)(bx_2 + cy_2) = ab(x_1x_2) + c(ax_1y_2 + bx_1y_2 + cy_1y_2)$. Hence, $(ab)x + cy = 1$ has integer solutions x and y . Hence, $\gcd(ab, c) = 1$.

7. (15 points) a) Use the Extended Euclidean Algorithm (E.E.A) to find a particular solution of the equation $57x + 12y = \gcd(57, 12)$. (Credit will be given only if the E.E.A is used).

$$57x + 12y = 12$$

x	y	r	ℓ
1	0	57	
0	1	12	
1	-4	9	4
-1	5	3	1
		0	

$$\gcd(57, 12)$$

$$57(-1) + 12 \cdot 5 = 3 = \gcd(57, 12)$$

- b) Find all the integer solutions of the equation $57x + 12y = 300$. Show all your work. $(x_0, y_0) = 100(-1, 5) = (-100, 500)$ is one particular

solution. The general sol'n of $57x + 12y = 0$

$$\Leftrightarrow 19x + 4y = 0$$

is $(x_n, y_n) = (+4k, -19k)$, $k \in \mathbb{Z}$. So the general sol'n of ④ is $(x, y) = (x_0 + x_n, y_0 + y_n) = (-100 + 4k, 500 - 19k)$, $k \in \mathbb{Z}$

- c) Find all positive integer solutions of the equation $57x + 12y = 300$.

$$-100 + 4k > 0 \Rightarrow k > 25$$

$$500 - 19k > 0 \Rightarrow k < \frac{500}{19} = 26.3 \dots < 27$$

Hence, $K = 26$ is the unique sol'n.

So $(x, y) = (4, 6)$ is the unique positive sol'n of ④.

8. (15 points) Use induction to prove the following inequality for all positive integers n .

$$S_m \stackrel{\text{def}}{=} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \right) \geq 1 + \frac{n}{2}.$$

Initial Step

Case $m = 1$: $\frac{1}{1} + \frac{1}{2} \geq 1 + \frac{1}{2}$ is true,

Induction Step: Assume \circledast holds for m ,

We need to show that

$$\underbrace{\left(\frac{1}{1} + \cdots + \frac{1}{2^m} \right)}_{S_m} + \underbrace{\left(\frac{1}{2^m+1} + \cdots + \frac{1}{2^{m+1}} \right)}_{S_{m+1} - S_m} \geq 1 + \frac{m+1}{2}.$$

Note that each summand in the second bracket above is larger than or equal to $\frac{1}{2^{m+1}}$ and there are 2^m summands. Hence,

$$S_{m+1} - S_m \geq 2^m \cdot \frac{1}{2^{m+1}} = \frac{1}{2}. \text{ Thus,}$$

$$S_{m+1} \geq S_m + \frac{1}{2} \stackrel{\substack{\text{Induction} \\ \text{Hypothesis}}}{\geq} \left(1 + \frac{m}{2} \right) + \frac{1}{2} = 1 + \frac{m+1}{2}.$$

Hence, the inequality \circledast holds for $m+1$.

The statement follows by the Principle of Mathematical Induction.