

1. (15 points) The matrices A and B below are row equivalent (you do **not** need to check this fact).

$$A = \begin{pmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) The rank of A is 3.

b) A basis for the null space of A : The reduced echelon form is $\begin{pmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

The free variables are x_2 and x_4 . The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_4 \\ x_2 \\ \frac{3}{2}x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} -5 \\ 0 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

and the two column vectors on the right hand side are a basis for $Null(A)$.

c) A basis for the column space of A : The first, third and fifth columns of A are its pivot columns. They are a basis for the column space of A .

d) A basis for the row space of A : Take the three non-zero rows of B .

2. (4 points) The null space of the 5×6 matrix A is 2 dimensional. What is the dimension of (a) the Row space of A ? (b) the Column space of A ? **Justify your answer!**

a) $\dim(Null(A)) + \dim(Row(A)) = 6$. Thus, $\dim(Row(A)) = 6 - 2 = 4$.

b) The dimension of the column space is equal to the dimension of the row space, which is 4.

3. (15 points)

(a) Use the cofactor expansion along the third column (which has two zero en-

tries) to calculate the characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 4 & -4 & -1 \end{pmatrix}$.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1 - \lambda & 1 & 0 \\ 0 & 2 - \lambda & 0 \\ 4 & -4 & -1 - \lambda \end{bmatrix} = -(1 + \lambda) \det \begin{bmatrix} 1 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = \\ &= -(\lambda + 1)(\lambda - 1)(\lambda - 2) \end{aligned}$$

(b) A basis of \mathbb{R}^3 consisting of eigenvectors of A :

The three eigenvalues are -1 , 1 and 2 . The -1 -eigenspace is $\text{Null}(A + I)$

and it is spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The 1 -eigenspace is $\text{Null}(A - I)$ and it is spanned by $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

The 2 -eigenspace is $\text{Null}(A - 2I)$ and it is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Hence, $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of eigenvectors of A .

(c) The 3×3 matrix $P = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, whose columns are the three eigenvectors above, satisfies

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

4. (12 points)

(a) $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ is diagonalizable because it is a symmetric matrix with real entries.

(b) The matrix $A = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$ has only one eigenvalue 2 . The 2 -eigenspace $\text{Null}(A - 2I) = \text{Null}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ is one-dimensional. Hence, the matrix A does not have two linearly independent eigenvectors. It is thus *not* diagonalizable.

(c) The characteristic polynomial of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is $\det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$. Hence, A does not have any real eigenvalue. In particular, A is not diagonalizable via matrices with real entries.

5. (22 points) Let W be the plane in \mathbb{R}^3 spanned by $v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(a) The length of v_1 is $\sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$

(b) The distance between the two points v_1 and v_2 in \mathbb{R}^3 is: $\|v_1 - v_2\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$.

(c) The subspace W^\perp orthogonal to W is:

$$W^\perp = \text{Null} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \text{Null} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}. \text{ It is spanned by } u := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ Hence, } \frac{1}{\|u\|}u = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ is a unit vector in } W^\perp.$$

(d) The projection of v_2 to the line spanned by v_1 is:

$$\frac{v_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 = \frac{3}{6} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

(e) Write v_2 as the sum of a vector parallel to v_1 and a vector orthogonal to v_1 : Subtract from v_2 its orthogonal projection to the line spanned by v_1 . The resulting vector

$$v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

will be perpendicular to the line spanned by v_1 . Thus,

$$v_2 = \left(\frac{v_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 \right) + \left(v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 \right) = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

is the required decomposition of v_2 .

(f) An orthogonal basis for W is provided by $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} \right\}$

6. (16 points) Let W be the plane in \mathbb{R}^3 spanned by $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(a) The vectors u_1 and u_2 provide an *orthogonal* basis for W . Hence, the projection of $b = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ to W is:

$$\hat{b} = \frac{b \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{b \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{3}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{3}{2} \end{pmatrix}$$

(b) The distance from b to W is

$$\|b - \hat{b}\| = \left\| \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\| = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \sqrt{\frac{3}{2}}$$

- (c) A least square solution to the equation $Ax = b$, where A is the 3×2 matrix with columns u_1 and u_2 , is a vector x in \mathbb{R}^2 which minimizes the length $\|Ax - b\|$. We can calculate it in two ways:

First Method: (using part 6a). The vector Ax will be in the subspace W . The point in W closest to b is \hat{b} . Solve $Ax = \hat{b}$. Since $\hat{b} = 1 \cdot u_1 + \frac{1}{2}u_2$, then

$$x = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}.$$

Second Method: We can calculate x directly as the solution of the equation

$$\begin{aligned} A^T Ax &= A^T b = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} x &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ x &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

- (d) Find the coefficients c_0, c_1 of the line $y(x) = c_0 + c_1x$ which best fits the three points $(x_1, y_1) = (-1, 0)$, $(x_2, y_2) = (0, 2)$, $(x_3, y_3) = (1, 1)$ in the x, y plane.

The line should minimize the sum $\sum_{i=1}^3 [y(x_i) - y_i]^2$.

The value $y(x_i) = c_0 + c_1x_i$ can be written as the dot product of $(1, x_i)$ with $\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$. Hence, the sum $\sum_{i=1}^3 [y(x_i) - y_i]^2$ can be written as the square of the norm of the vector

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Plugging in the coordinates (x_i, y_i) of the three points given, we get

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

The problem reduces to part 6c with $\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = x = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$

7. (16 points) The vectors $v_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .8 & .5 \\ .2 & .5 \end{pmatrix}$.

- (a) We calculate $Av_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = v_1$. Hence, the eigenvalue of v_1 is 1.

The eigenvalue of v_2 is .3 because $Av_2 = \begin{pmatrix} .3 \\ -.3 \end{pmatrix} = .3v_2$

- (b) The coordinates c_1, c_2 of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the basis $\{v_1, v_2\}$ are its coefficients as a linear combination $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1v_1 + c_2v_2$. We find them by row reduction

$$\begin{pmatrix} 5 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 1 & \frac{2}{7} \end{pmatrix}$$

So, $c_1 = \frac{1}{7}$ and $c_2 = \frac{2}{7}$.

- (c) $A^{100} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A^{100}(\frac{1}{7}v_1 + \frac{2}{7}v_2) = (1)^{100}\frac{1}{7}v_1 + (.3)^{100}\frac{2}{7}v_2 = \begin{pmatrix} \frac{5}{7} + \frac{2}{7}(.3)^{100} \\ \frac{2}{7} - \frac{2}{7}(.3)^{100} \end{pmatrix}$.
- (d) $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1)^n\frac{1}{7}v_1 + (.3)^n\frac{2}{7}v_2$. As n gets larger, the vector $(.3)^n\frac{2}{7}v_2$ approaches the zero vector. Hence, as n gets larger, the vector $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ approaches $\frac{1}{7}v_1$ which is equal to $\begin{pmatrix} \frac{5}{7} \\ \frac{2}{7} \end{pmatrix}$.