Math 235 section 1 Midterm 1 Spring 2014

Your Name: My Solution

Student ID: ______________________

This is a 90 minutes exam. This exam paper consists of 6 questions. It has 7 pages.

The use of calculators is not allowed on this exam. You may use one letter size page of notes (both sides), but no books.

It is not sufficient to just write the answers. You must *explain* how you arrive at your answers.

1. (20)	______
2. (15)	______
3. (20)	______
4. (10)	______
5. (20)	______
6. (15)	______
TOTAL (100)	

12 pts

1. (20 points) a) Show that the row **reduced** echelon form of the augmented matrix of the system

$$\begin{array}{rcl} x_1 \qquad + x_3 - x_4 + x_5 &=& 1 \\ 3x_1 + 2x_2 + x_3 - 3x_4 - x_5 &=& 1 \\ x_1 + x_2 \qquad - x_4 + x_5 &=& 2 \end{array} \quad \text{is} \quad \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

Use at most seven elementary operations. Show all your work. Clearly write in words each elementary row operation you used.

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 1 & 1 \\ 3 & 2 & 1 & -3 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 2 \end{array}\right) \underset{\text{Add } -R_1 \text{ to } R_3}{\overset{\text{Add } -3R_1 \text{ to } R_2}{\sim}} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 2 & -2 & 0 & -4 & -2 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \underset{\text{Add } -R_2 (\text{new}) \text{ to } R_3}{\overset{\text{Multiply } R_2 \text{ by } \frac{1}{2}}{\sim}}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{array}\right) \begin{array}{l} \text{Add } -\frac{1}{2}R_3 \text{ to } R_1 \\ \sim \text{Add } R_3 \text{ to } R_2 \\ \text{Multiply } R_3 \text{ by } \frac{1}{2} \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

8 pts

b) Find the general solution for the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 + x_4 \\ x_3 + 1 \\ x_3 \\ x_4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$x_3 \quad x_4$

2. (15 points) You are given that the row reduced echelon form of the matrix $A = \begin{pmatrix} 2 & 1 & 1 & -2 & 2 \\ 3 & 2 & 1 & -3 & -1 \\ 1 & 1 & 0 & -1 & 1 \end{pmatrix}$ is $B = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. You do **not** need to verify it.

5 pt

(a) Write the general solution of the system $A\vec{x} = \vec{0}$ in parametric form $\vec{x} =$ (first free variable)$\vec{v}_1 +$ (second free variable)$\vec{v}_2 + \cdots$

x_3, x_4 are free variables. $x_1 = -x_3 + x_4$, $x_2 = x_3$, $x_5 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 + x_4 \\ x_3 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

5 pt

(b) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$. Find a finite set of vectors in $\ker(T)$ which spans $\ker(T)$.

Every vector in $\ker(T)$ is a linear combination of $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, by part (a), hence these two vectors span $\ker(T)$.

5 pts

(c) Is the image of T equal to the whole of $\mathbb{R}^3$? **Justify your answer!**

Yes, A has a pivot position in every row, hence, $(A|\vec{b})$ can not have a pivot in the rightmost column, hence $A\vec{x} = \vec{b}$ is consistent, for every $\vec{b}$ in $\mathbb{R}^3$. Hence, every vector $\vec{b}$ in $\mathbb{R}^3$ is in the image of T.

5 pts

3. a) (13 points) Determine for which values of k the 3×3 matrix $A = \begin{pmatrix} 1 & 1 & 2+k \\ 1 & 2 & 3+k \\ 1 & 3 & 5+k \end{pmatrix}$ is invertible and find the inverse (expressed in terms of the parameter k) for **all** values of k for which it exists.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2+k & 1 & 0 & 0 \\ 1 & 2 & 3+k & 0 & 1 & 0 \\ 1 & 3 & 5+k & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2+k & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right) \sim$$

Add $-R_2$ to R_3

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2+k & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -k-1 & 4+2k & -k-2 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \sim$$

Add $-R_2$ to R_1

Add $-R_3$ to R_2

Add $(-2-k)R_3$ to R_1

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1-k & (1+2k) & (-k-1) \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

Invertible for all values of k, $A^{-1} = \begin{pmatrix} 1-k & (1+2k) & (-1-k) \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$

b) (2 points) Use matrix multiplication to check that the matrix you found is indeed A^{-1}.

$$\begin{pmatrix} 1 & 1 & 2+k \\ 1 & 2 & 3+k \\ 1 & 3 & 5+k \end{pmatrix} \begin{pmatrix} 1-k & (1+2k) & (-1-k) \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) (5 points) Let A, B, C, D be $n \times n$ matrices, with A and B invertible, which satisfy the equation $ABCB^{-1}A^{-1} = D$. Express C in terms of A, B, and D. Show all your work.

Multiply both sides by A^{-1} on the left and A on the right to get:

$$BCB^{-1} = A^{-1}DA$$

Repeat with B^{-1} on the left and B on the right to get:

$$\boxed{C = B^{-1}A^{-1}DAB}$$

4. (10 points) Consider a 3×4 matrix A and a 4×5 matrix B. If $\ker(A) = \operatorname{im}(B)$, what can you say about the matrix AB? **Justify your answer.**

$$(AB)\vec{x} = A(\underbrace{B\vec{x}}_{\text{in } \operatorname{im}(B) = \ker(A)}) = \vec{0}$$

Hence, $(AB)\vec{x} = \vec{0}$ for all $\vec{x}$ in $\mathbb{R}^5$

Hence $\underset{3\times 5}{AB}$ is the zero 3×5 matrix.

$$AB = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

5. (20 points) Let L be the line in $\mathbb{R}^2$ through the origin and the vector $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
Recall that the reflection $Ref_L : \mathbb{R}^2 \to \mathbb{R}^2$ is given by the formula

$$Ref_L(\vec{x}) \quad = \quad \frac{2(\vec{x}\cdot\vec{v})}{\vec{v}\cdot\vec{v}}\vec{v} - \vec{x}. \tag{1}$$

8 pts

(a) Use the formula (1) to find the standard matrix A of Ref_L.

$$\vec{a}_1 = A\begin{pmatrix}1\\0\end{pmatrix} = Ref_L\begin{pmatrix}1\\0\end{pmatrix} = \frac{2\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}}{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}}\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \frac{4}{5}\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3/5\\4/5\end{pmatrix}$$

$$\vec{a}_2 = A\begin{pmatrix}0\\1\end{pmatrix} = Ref_L\begin{pmatrix}0\\1\end{pmatrix} = \frac{2\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}}{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}}\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \frac{2}{5}\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}4/5\\-3/5\end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix}3/5 & 4/5\\4/5 & -3/5\end{pmatrix}}_{=(\vec{a}_1\ \vec{a}_2)}$$

(b) Let M be the line in $\mathbb{R}^2$ through the origin and the vector $\vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. The matrix of the reflection $Ref_M : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the line M is $B := \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & -4 \end{pmatrix}$. You do **not** need to verify it. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the composition $T(\vec{x}) = Ref_L(Ref_M(\vec{x}))$. Express the standard matrix C of T in terms of the matrices A and B. $C =$ $A \cdot B$

Use the expression above to show that $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$$C = \frac{1}{5}\begin{pmatrix}3 & 4\\4 & -3\end{pmatrix}\frac{1}{5}\begin{pmatrix}4 & -3\\-3 & -4\end{pmatrix} = \frac{1}{25}\begin{pmatrix}0 & -25\\25 & 0\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$$

6 pts

(c) Interpret the matrix C geometrically.

$$C = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}\cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2})\\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2})\end{pmatrix}$$ is the matrix of the rotation of $\mathbb{R}^2$ $\frac{\pi}{2}$ radians ($90°$) counterclockwise.

6

6. (15 points)

8 pts

(a) Find all the values of k for which the vector $\begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix}$ is a linear combination of the vectors $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$? Justify your answer! These are the values of k for which the vector equation $x_1\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix}$ is consistent.

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & -1 & k \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & k-2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & k+4 \end{array}\right)$$

Add R_1 to R_2
Add $-2R_1$ to R_3

Add $3R_2$ to R_3

The linear system is consistent, if and only if we do not have a pivot in the rightmost column, if and only if $\boxed{k=-4}$

7 pt

(b) Let A be a 3×2 matrix such that the system $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ has a unique solution. What is the rank of A? Justify your answer!

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

A must have a pivot in every column, for the system to have a unique sol'n.

Hence $\text{rank}(A) = 2$.