Math 235 section 1 Midterm 1 Spring 2014

1. (20 points) a) Show that the row **reduced** echelon form of the augmented matrix of the system $\begin{array}{rcl} x_1 \quad\quad + x_3 - x_4 + x_5 & = & 1 \\ 3x_1 + 2x_2 + x_3 - 3x_4 - x_5 & = & 1 \\ x_1 + x_2 \quad\quad - x_4 + x_5 & = & 2 \end{array}$ is $\begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Use at most seven elementary operations. Show all your work. Clearly write in words each elementary row operation you used.

   b) Find the general solution for the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} =$$

2. (15 points) You are given that the row reduced echelon form of the matrix $A = \begin{pmatrix} 2 & 1 & 1 & -2 & 2 \\ 3 & 2 & 1 & -3 & -1 \\ 1 & 1 & 0 & -1 & 1 \end{pmatrix}$ is $B = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. You do **not** need to verify it.

   (a) Write the general solution of the system $A\vec{x} = \vec{0}$ in parametric form $\vec{x} = (\text{first free variable})\vec{v}_1 + (\text{second free variable})\vec{v}_2 + \cdots$

   (b) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$. Find a finite set of vectors in $\ker(T)$ which spans $\ker(T)$.

   (c) Is the image of T equal to the whole of $\mathbb{R}^3$? **Justify your answer!**

3. a) (13 points) Determine for which values of k the 3×3 matrix $A = \begin{pmatrix} 1 & 1 & 2+k \\ 1 & 2 & 3+k \\ 1 & 3 & 5+k \end{pmatrix}$ is invertible and find the inverse (expressed in terms of the parameter k) for **all** values of k for which it exists.

   b) (2 points) Use matrix multiplication to check that the matrix you found is indeed A^{-1}.

   c) (5 points) Let A, B, C, D be $n \times n$ matrices, with A and B invertible, which satisfy the equation $ABCB^{-1}A^{-1} = D$. Express C in terms of A, B, and D. Show all your work.

4. (10 points) Consider a 3×4 matrix A and a 4×5 matrix B. If $\ker(A) = \text{im}(B)$, what can you say about the matrix AB? **Justify your answer.**

5. (20 points) Let L be the line in $\mathbb{R}^2$ through the origin and the vector $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Recall that the reflection $Ref_L : \mathbb{R}^2 \to \mathbb{R}^2$ is given by the formula

$$Ref_L(\vec{x}) = \frac{2(\vec{x} \cdot \vec{v})}{\vec{v} \cdot \vec{v}}\vec{v} - \vec{x}. \quad (1)$$

(a) Use the formula (1) to find the standard matrix A of Ref_L.

(b) Let M be the line in $\mathbb{R}^2$ through the origin and the vector $\vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. The matrix of the reflection $Ref_M : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the line M is $B := \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -3 & -4 \end{pmatrix}$. You do **not** need to verify it. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the composition $T(\vec{x}) = Ref_L(Ref_M(\vec{x}))$. Express the standard matrix C of T in terms of the matrices A and B. $C =$____.

Use the expression above to show that $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(c) Interpret the matrix C geometrically.

6. (15 points)

(a) Find all the values of k for which the vector $\begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix}$ is a linear combination of the vectors $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$? Justify your answer!

(b) Let A be a 3×2 matrix such that the system $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ has a unique solution. What is the rank of A? Justify your answer!