Math 235 Midterm 2 Spring 2014

Your Name: SOLUTION

Student ID: ____________________

Your Instructor's Name: ____________________

This is a two hours exam. This exam paper consists of 7 questions. It has 9 pages.

The use of calculators is not allowed on this exam. You may use one letter size page of notes (both sides), but no books.

It is not sufficient to just write the answers. You must *explain* how you arrive at your answers.

1. (18)	
2. (12)	
3. (16)	
4. (12)	
5. (14)	
6. (14)	
7. (14)	
TOTAL (100)	

1. (18 points) You are given below the matrix A together with its row reduced echelon form B

$$A = \begin{pmatrix} 1 & -1 & -3 & -3 & 0 & -3 \\ 1 & 0 & 2 & 3 & 0 & 4 \\ 2 & 0 & 4 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{pmatrix} \qquad B = \begin{array}{c} \begin{array}{cccccc} & & x_3 & x_4 & & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 2 & 3 & 0 & 4 \\ 0 & 1 & 5 & 6 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Note: you do **not** have to check that A and B are indeed row equivalent.

a) Determine the rank of A. Explain how it is determined by the matrix B.

$\operatorname{rank}(A) = 3$ = number of pivot positions in B

b) Find a basis for the kernel $\ker(A)$ of A. Free var x_3, x_4, x_6

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2x_3 - 3x_4 - 4x_6 \\ -5x_3 - 6x_4 - 7x_6 \\ x_3 \\ x_4 \\ -8x_6 \\ x_6 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + x_4 \underbrace{\begin{pmatrix} -3 \\ -6 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} + x_6 \underbrace{\begin{pmatrix} -4 \\ -7 \\ 0 \\ 0 \\ -8 \\ 1 \end{pmatrix}}_{v_3}$$

$\{v_1, v_2, v_3\}$ is a basis.

c) Find a basis for the image $\operatorname{im}(A)$ of A. Justify your answer

The pivot columns of A form a basis. These are

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad a_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

d) Let β be the basis you found in part 1c for the image of A and let $\vec{a}_6$ be the sixth column of A. Find the β-coordinate vector $[\vec{a}_6]_\beta$ of $\vec{a}_6$.

$\vec{b}_6 = 4\vec{b}_1 + 7\vec{b}_2 + 8\vec{b}_5$, so $\vec{a}_6 = 4\vec{a}_1 + 7\vec{a}_2 + 8\vec{a}_5$,

$[\vec{a}_6]_\beta = \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$.

2. (12 points) For which values of the constant k do the vectors below form a basis of $\mathbb{R}^3$. Justify your answer!

$$\underset{v_1}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}, \begin{pmatrix} -1 \\ 7 \\ k \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 7 \\ 1 & 1 & k \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 9 \\ 0 & -1 & k+1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & k-2 \end{pmatrix}$$

If $k \neq 2$, the matrix has rank 3, so the thee vectors are linearly independent (pivot in every column) and span $\mathbb{R}^3$ (pivot in every row). If $k = 2$, $v_3 = 5v_1 - 3v_2$, and

$\{v_1, v_2, v_3\}$ is linearly dependent

3. (16 points) Let $\vec{v}_1$ be a non-zero vector in $\mathbb{R}^2$. Recall that the reflection $T : \mathbb{R}^2 \to \mathbb{R}^2$, with respect to the line spanned by $\vec{v}_1$, is given by

$$T(\vec{x}) = 2\left(\frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 - \vec{x}. \tag{1}$$

(a) Let $\beta := \{\vec{v}_1, \vec{v}_2\}$ be a basis of $\mathbb{R}^2$ such that $\vec{v}_1 \cdot \vec{v}_2 = 0$ (the two vectors are orthogonal). Let T be the reflection with respect to the line spanned by $\vec{v}_1$. Express $T(\vec{v}_1)$ and $T(\vec{v}_2)$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

$$T(\vec{v}_1) = 2\underbrace{\left(\frac{\vec{v}_1\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)}_{1}\vec{v}_1 - \vec{v}_1 = \vec{v}_1$$

$$T(\vec{v}_2) = 2\left(\frac{0}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 - \vec{v}_2 = -\vec{v}_2$$

(b) Use your calculations in part 3a to find the β-matrix B of T.

$$B = \left([T(v_1)]_\beta \; [T(v_2)]_\beta\right) = \left(\underbrace{[v_1]_\beta}_{\begin{pmatrix}1\\0\end{pmatrix}} \; \underbrace{[-v_2]_\beta}_{\begin{pmatrix}0\\-1\end{pmatrix}}\right) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

(c) Assume from now on that T is the reflection with respect to the line spanned by $\vec{v}_1 = \begin{pmatrix}2\\3\end{pmatrix}$. Find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$.

$$T(\vec{e}_1) = 2\left(\frac{\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}}{\underbrace{\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}}_{4+9=13}}\right)\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \frac{4}{13}\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-5/13\\12/13\end{pmatrix}$$

$$T(\vec{e}_2) = 2\left(\frac{\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}}{13}\right)\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \frac{6}{13}\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}12/13\\5/13\end{pmatrix}$$

(d) Use your work in part 3c to show that the matrix of T with respect to the standard basis $\{\vec{e}_1, \vec{e}_2\}$ is $A = \frac{1}{13}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}$.

$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{pmatrix}$$

(e) Let $\beta = \{\vec{v}_1, \vec{v}_2\}$ be the basis of $\mathbb{R}^2$, where the vector $\vec{v}_1$ is given in part 3c and $\vec{v}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Note that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal $\vec{v}_1 \cdot \vec{v}_2 = 0$. Find a matrix S, such that $S^{-1}AS$ is equal to the β-matrix B of T you found in part 3b, where A is the standard matrix you found in part 3d.

$$S = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$$

(f) Explicitly verify that the matrices A, B, and S in part 3e satisfy the equality $SB = AS$, by calculating each side.

$$\underbrace{\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}} = \frac{1}{13}\underbrace{\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}}$$

$$\frac{1}{13}\begin{pmatrix} 26 & -39 \\ 39 & 26 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

4. (12 points) Let A be a 5×4 matrix with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$. We are given that the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ belongs to the kernel of A and the vectors $\underbrace{\begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}}_{u}$ and $\underbrace{\begin{pmatrix} 6 \\ 7 \\ 8 \\ 9 \\ 0 \end{pmatrix}}_{v}$ span the image of A.

(a) Express $\vec{a}_4$ as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

$1 \cdot \vec{a}_1 + 2\vec{a}_2 + 3\vec{a}_3 + 4\vec{a}_4 = \vec{0}$, so

$$\vec{a}_4 = -\frac{1}{4}\vec{a}_1 - \frac{2}{4}\vec{a}_2 - \frac{3}{4}\vec{a}_3$$

(b) Determine the dimension of the image of A. Justify your answer.

The vectors u, v span the image of A and are clearly linearly indep (u is not a scalar mult of v). Hence, they form a basis of $\operatorname{im}(A)$ consisting of two vectors. Thus, $\dim(\operatorname{im}(A)) = 2$.

(c) Determine the dimension of the kernel of A. Justify your answer.

$4 = \dim(\ker(A)) + \underbrace{\dim(\operatorname{im}(A))}_{=2}$, by the rank-nullity theorem. Thus, $\dim(\ker(A)) = 4 - 2 = 2$.

5. (14 points) Let P_2 be the space of all polynomials $a_0 + a_1 t + a_2 t^2$ of degree ≤ 2. Find a basis for the subspace W of P_2 consisting of all polynomials $f(t)$ satisfying $f'(1) = 0$. Explain why the set you found spans W and why it is linearly independent.

$$f(t) = a_0 + a_1 t + a_2 t^2$$

$$f'(t) = a_1 + 2a_2 t$$

$$f'(1) = a_1 + 2a_2$$

The equation $f'(1)=0$ yields the linear equation

$$0 \cdot a_0 + 1 \cdot a_1 + 2 \cdot a_2 = 0$$

The coefficients ("variables") a_0, a_2 are free and

$$a_1 = -2a_2.$$

So $f(t) = a_0 + (-2a_2)t + a_2 t^2 = a_0 \cdot \underbrace{1}_{\text{the constant poly } 1} + a_2(-2t+t^2)$

$\{1, -2t+t^2\}$ is a basis for W.

These two polynomials span W, since a polynomial $f(t)$ in W was shown above to be a linear combination of them.

These poly are linearly independent, since if

$c_1 \cdot 1 + c_2(-2t+t^2)$ is the constant zero poly,

then its constant term c_1 and the coeff c_2 of t^2 must both be zero.

6. (14 points) Determine which of the following subsets is a subspace by verifying the properties in the definition of a subspace or by showing that one of those properties does not hold.

(a) The subset W of all 2×2 matrices A satisfying $AB = BA$, where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. W is a subspace.

(1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in W, since $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = B \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(2) If A and C are in W, then $AB \overset{(*)}{=} BA$ and $CB \overset{(**)}{=} BC$.

So $(A+C)B = AB + CB \overset{(*)+(**)}{=} BA + BC = B(A+C)$. Hence, $A+C$ is in W.

(3) If A is in W and c is a scalar, then

$(cA)B = c(AB) \overset{(*)}{=} c(BA) = B(cA)$.

So cA is in W.

(b) The subset W of $\mathbb{R}^4$ consisting of vectors of the form $\begin{pmatrix} x-y \\ y-z \\ x+z \\ y \end{pmatrix}$, where x, y, z are arbitrary real numbers.

$$= x\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + z\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

W is a subspace.

$$W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

We know that the span of a set of vectors in a vector space is a subspace (satisfies properties (1), (2), (3) in the def),

7. (14 points)

(a) Consider a matrix A and let B be the row reduced echelon form of A. Explain why the statement is true or provide a counter example.

True:

i. Is $\ker(A)$ necessarily equal to $\ker(B)$?

A and B are row equivalent, so the set of solutions of $A\vec{x}=\vec{0}$ is equal to the set of solutions of $B\vec{x}=\vec{0}$. The former set is $\ker(A)$ and the latter is $\ker(B)$, by definition.

ii. Is the image of A necessarily equal to the image of B?

False! Take $A=\begin{pmatrix}1&2\\2&4\end{pmatrix}$. Then $B=\text{rref}(A)=\begin{pmatrix}1&2\\0&0\end{pmatrix}$

$\text{im}(A)=\text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$, $\text{im}(B)=\text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$,

This are two different lines in $\mathbb{R}^2$.

(b) Let A be a 4×3 matrix and B a 3×4 matrix. Show that $\text{rank}(AB)\le 3$. Hint: Relate $\text{im}(AB)$ and $\text{im}(A)$?

AB is a 4×4 matrix.

If $\vec{y}$ belongs to $\text{im}(AB)$, then $\vec{y}=(AB)\vec{x}$ for some $\vec{x}$. Then $\vec{y}=A(B\vec{x})$ is a value of A as well, so $\vec{y}$ belongs to $\text{im}(A)$. Thus, $\text{im}(AB)$ is contained in $\text{im}(A)$. We conclude that

$$\boxed{\dim(\text{im}(AB))\le\dim(\text{im}(A))}$$ Thus

$\text{rank}(AB)=\dim(\text{im}(AB))\le\dim(\text{im}(A))=\text{rank}(A)$.

Now A has 3 columns, so $\text{rank}(A)\le 3$.

We conclude that $\text{rank}(AB)\le 3$.