MATH 235 SPRING 2011
FINAL EXAM

1. (18 points)

   (a) Consider the complex plane $\mathbb{C}$ as a two dimensional vector space with basis $\beta = \{1, i\}$. Let $T : \mathbb{C} \to \mathbb{C}$ be multiplication by the complex number $2 + 3i$, i.e., $T(z) = (2 + 3i)z$. Find the β-matrix of T.

   (b) Let $A = \begin{pmatrix} 5 & -5 \\ 4 & 1 \end{pmatrix}$. Find the characteristic polynomial of A and determine the eigenvalues of A.

   (c) Find an invertible matrix P, with *complex* entries, and a diagonal matrix D, such that $P^{-1}AP = D$. Justify your answer!

   (d) Find an invertible matrix S, with *real* entries, and real numbers a, b, such that $S^{-1}AS = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Justify your answer.

2. (18 points)

   (a) Assume given a 3×3 matrix A and a 3×3 upper triangular matrix $U = \begin{pmatrix} 2 & u_{12} & u_{13} \\ 0 & 3 & u_{23} \\ 0 & 0 & 5 \end{pmatrix}$. Consider the sequence of row operations
   1) Interchange row 1 and row 2 of A to obtain the matrix B.
   2) Multiply by $\frac{1}{2}$ row 3 of B to obtain the matrix C.
   2) Add -2 times row 1 to row 2 of C to obtain the matrix D.
   3) Add row 1 to row 3 of D to obtain the matrix E.
   4) Add -3 times row 2 to row 3 of E to obtain the matrix U.
   Assume that these elementary row operations reduce A to U. Compute $\det(A)$. Justify your answer!

   (b) For which values of the real constants a and b is the matrix $\begin{pmatrix} 2 & a \\ 0 & b \end{pmatrix}$ diagonalizable? Justify your answer!

   (c) Let $\mathbb{R}^{3\times 3}$ be the vector space of matrices of size 3×3 and $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^4$ a linear transformation. What are all the possible values of $\dim(\ker(T))$? Justify your answer!

3. (a) (5 points) Find *all* orthogonal matrices of the form $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & a \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & b \\ \frac{1}{\sqrt{3}} & 0 & c \end{pmatrix}$.

   (b) (5 points) Let A be an $n \times n$ matrix and A^T its transpose. Recall that $\det(A) = \det(A^T)$ and $\det(AB) = \det(A)\det(B)$ for any $n \times n$ matrix B. Use the above properties of the determinant to show that if A is an orthogonal $n \times n$ matrix, then $\det(A)$ is equal to 1 or -1.

4. (18 points) The vectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$.

(a) The eigenvalue of v_1 is ______
The eigenvalue of v_2 is ______

(b) Set $w := \begin{pmatrix} 13 \\ 13 \end{pmatrix}$. Find the coordinate vector $[w]_\beta$ of w in the basis $\beta := \{v_1, v_2\}$.

(c) Compute $A^{100} \begin{pmatrix} 13 \\ 13 \end{pmatrix}$.

(d) As n gets larger, the vector $A^n \begin{pmatrix} 13 \\ 13 \end{pmatrix}$ approaches ____. Justify your answer.

5. (18 points)

(a) Let P be the vector space of polynomials of arbitrary degree. Consider the transformation $T : P \to P$, given by $T(f(t)) = t^2 f'(t) - 2tf(t) + 2f''(t)$. Show that T is linear.

(b) P_2 the subspace of P of polynomials of degree ≤ 2. Note that T maps P_2 into P_2. Let $S : P_2 \to P_2$ be given by the same formula above, $S(f(t)) = t^2 f'(t) - 2tf(t) + 2f''(t)$. Find the matrix of S in the basis $\beta = \{1, t, t^2\}$.

(c) Determine if S is an isomorphism. Justify your answer!

(d) The function $f(t) = t^2 - 2t + 2$ is an eigenvector of S. What is its eigenvalue? Justify your answer!

6. (18 points) Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$, and V the subspace of $\mathbb{R}^4$ spanned by v_1 and v_2.

(a) Let $w = \begin{pmatrix} 20 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Find the orthogonal projection $Proj_V(w)$ of w to V. Justify your answer!

(b) Write w as a sum of a vector in V and a vector orthogonal to V.

(c) Find the distance from w to V, i.e., the distance from w to the vector in V closest to w.

(d) Let W be the subspace of $\mathbb{R}^4$ spanned by the set $\beta := \{v_1, v_2, w\}$. Use the Gram-Schmidt process with the basis β of W to find an orthonormal basis of W. Explain every step of the Gram-Schmidt process you used.