Practice Problems Math 235 Fall 2010 for Midterm on October 28

1. a: Solve the system of equations using row operations.

$$x + y - z = 6$$

$$2x - y = 0$$

$$3x - y - 2z = -3$$

   b: Write the above system of equations as a matrix equation.

   c: For what vectors $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ does the equation $Ax = v$ have a solution if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$, and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

   d: What is the rank of the matrix A.

   e: What is the dimension of the image of A? What is the dimension of $ker(A)$?

2. a: Define what it means for a function $F : \mathbb{R}^m \to \mathbb{R}^n$ to be a linear transformation.
   b: Are the following linear transformations? Why? Note that the why part of the question is very important.
   b1: $F : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y - 1 \\ 3x - y \end{pmatrix}$
   b2: $F : \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x)$

   b3: $F : \mathbb{R}^3 \to \mathbb{R}, \vec{v} \mapsto (-1, 2, 3) \cdot v$.

3. Define the following terms:
   a: independence
   b: spans a subspace
   c: subspace
   d: kernel
   e: image
   f: dimension
   g: rank.
   You should say what kind of object each term applies to. For example we say "the rank *of a matrix*" or "of a linear map".

4. : Find the matrix of reflection of $\mathbb{R}^2$ about the line $y = 2x$.

5. True or False. You must explain the reason for your answer.

   a: Let M be a matrix. If the kernel of M is just the zero vector, then the columns of M are linearly independent.

b: If $u, v, w \in V, V$ a subspace of $\mathbb{R}^n$, then the vector $2u - 3v + 4w$ is also an element of V.

c: Assume that $\{v_1, v_2, \cdots, v_t\}$ is an independent set in $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. The set $\{T(v_1), T(v_2), \ldots, T(v_t)\}$ is also independent.

d: There exists a 3×3 matrix M so that $ker(M) = im(M)$.

e: If the kernel a a matrix B consists of the $\mathbf{0}$-vector alone, then the column vectors of B are independent.

6. Let A be the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 3 & 1 \end{pmatrix}.$$

a: Find a basis of the kernel of A.
b: Find a basis of the image of A.
c: What is the dimension of the kernel of A? What is the dimension of the image of A? Are your answers compatible with the rank nullity theorem? Explain.

7. Find a basis of the plane through the origin and orthogonal to

$$\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}.$$

8. Let

$$a = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \; c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Express

$$d = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

as a linear combination of a, b, c.

9. Find a basis of the kernel and image of the following linear transformations:
a: Orthogonal projection of $\mathbb{R}^2$ to the line $y = 5x$,
b: $T : \mathbb{R}^3 \to \mathbb{R}, \vec{v} \mapsto (1, 2, 3) \cdot \vec{v}$,
c: Rotation about the origin of the plane by angle $\pi/6$.

10. Find a basis of the space of vectors in $\mathbb{R}^3$ orthogonal to both $(-1, 1, 2)$ and $(2, 1, 0)$.

11. A town has two baseball teams, one named R and one named Y. Each season 60% of the fans of R stick with R and 40% switch to Y. Each season 80% of the fans of Y stick with Y and 20% switch to R. The total number of fans stays constant from

season to season. Let $r(n), y(n)$ denote the number of fans of R, Y during season n. Find a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that

$$\begin{pmatrix} r(n+1) \\ y(n+1) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r(n) \\ y(n) \end{pmatrix}.$$

Assume $r(0) = 100, y(0) = 50$. Compute, using M, the number of fans in season 1 of R, Y, that is, compute $r(1), y(1)$. Compute $r(2), y(2)$.

12. Find the 3×3 matrix that represents rotation of $\mathbb{R}^3$ by angle ϕ about the y-axis.

13. Let $f : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear map. We are given that

$$f(u) = (0, 1, 2, 3, 4)$$
$$f(v) = (-1, 2, 6, 1, 4).$$

What is $f(2u - 3v)$?

14. Which of the following are subspaces of the indicated space. Explain your answer.
a: The set of of solutions in $\mathbb{R}^3$ to the equation $3x - y + 2z = 1$.
b: The set of vectors in $\mathbb{R}^4$ orthogonal to $(1, 2, 3, -4)$.

15. The matrix M of size 4×6 has kernel with dimension 2. How many independent column vectors does M have? Why? What is the dimension of the image of M? Why ?

16. Let

$$A = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & -4 \\ 4 & 7 \end{pmatrix}.$$

Find a 2×2 matrix X so that

$$AX = B.$$

17. Let

$$u = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

a: What vectors

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

are linear combinations of u, v?
b: Let

$$m = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

What vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are in the image of m.

c: Keep m from part b of this question. For what d, e, f can we solve the equation

$$mX = \begin{pmatrix} d \\ e \\ f \end{pmatrix}?$$