Practice Problems Math 235 Fall 2010 for Final Exam

1. a: Let S be a subset of a subspace W of a vector space V. What does it mean for S to be a basis of the subspace W..

   b: Is $\{1, (t-1), (t-1)^2, (t-1)^3\}$ a basis of P_3. Here P_3 denotes the vector space of all polynomials of degree less than or equal to 3. Why? Note that explaining why is the important part of the question.

2. Let T be the linear transformation from P_2 to P_2 given by

$$f(x) \mapsto f'' - 2f.$$

   Find the matrix of T with respect to the basis $\{1, x-1, (x-1)^2\}$.

3. False or True. (Please justify your answer with a counter example if false; if true, then explain why it is true.)

   (a) Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and assume that $\ker(L) = \{0\}$. Then for every $y \in \text{im}(L)$ there is a solution to the equation $L(x) = yandthesolutionisunique.$

   (b) The rank of a linear map $\mathbb{R}^{2011} \to \mathbb{R}^{2010}$ is at most 2010.

   (c) There is a linear map from $\mathbb{R}^{2011} \to \mathbb{R}^{2010}$ whose kernel is $\{0\}$.

   (d) Any set of 2010 vectors in $\mathbb{R}^{2011}$ must be linearly independent.

   (e) Reflection about a line in the plane is a linear map represented by a matrix of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where a, b are real numbers satisfying $a^2 + b^2 = 1$.

   (f) Any set of orthonormal (with respect to the usual "dot product") vectors in $\mathbb{R}^n$ is linearly independent.

   (g) Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and let $y \in \mathbb{R}^m$. Assume that $v \in \ker(L)$ is a solution to $L(x) = 0$ and that w is a solution to $L(x) = y$. Then $v + w$ is a solution to $L(x) = y$.

   (h) Suppose V is the vector space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. The map $F : V \to V : f(x) \mapsto f''(x) - 3f'(x) + 6f(x) - 9$ is linear.

   (i) Every 3×3 matrix has a real eigenvalue and eigenvector.

4. Let P_2 be the vector space of quadratic polynomials with standard basis $S = \{1, t, t^2\}$, and let $T : P_2 \to P_2 : p(t) \mapsto p'(t) + p(t)$.

   (a) Verify that T is a linear map.

   (b) Compute the matrix $[T]_{SS}$ for T with respect to the basis S.

   (c) Is T an isomorphism? (Why or why not?)

   (d) Find all polynomials $p(t)$ such that $T(p(t)) = 1 + t + t^2$.

5. Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis, and let $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\}$ be another basis of $\mathbb{R}^2$.

   (a) Suppose $v \in \mathbb{R}^2$ has coordinates $[v]_S = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ with respect to the standard basis S. What are its coordinates $[v]_B$ with respect to B?

   (b) If a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ has matrix $[L]_{SS} = \begin{pmatrix} 9 & -8 \\ 10 & -9 \end{pmatrix}$ in the standard basis S, what is its matrix $[L]_{BB}$ in the basis B?

6. Compute the determinant of the matrix $B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 5 & 6 & 0 \\ 0 & 7 & 8 & 0 \\ 3 & 0 & 0 & 4 \end{pmatrix}$.

   Find $\det(B^{235})$. If B is invertible, find $\det(B^{-1})$.

7. Let $C = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$.

   (a) Compute the characteristic polynomial of C.

   (b) Find the eigenvalues of C and an eigenvector for each eigenvalue.

   (c) Is C diagonalizable, that is, can we find an invertible matrix E so that $D = E^{-1}CE$ is diagonal? If so, find such matrices E and D; if not, explain why not.

8. Let $C = \begin{pmatrix} -7 & 3 \\ -18 & 8 \end{pmatrix}$.

   (a) Compute the characteristic polynomial of C.

   (b) Find the eigenvalues of C and an eigenvector for each eigenvalue.

   (c) Is C diagonalizable, that is, can we find an invertible matrix E so that $D = E^{-1}CE$ is diagonal? If so, find such matrices E and D; if not, explain why not.

9. Let

$$A = \begin{pmatrix} 1/3 & 10/3 \\ -4/3 & 5/3 \end{pmatrix}.$$

   (a) Compute the characteristic poynomial of A.

   (b) Find the eigenvalues and eigenvectors of A.

   (c) The eigenvalues are not real. Find a basis B of $\mathbb{R}^2$ so that the matrix of A with respect to the basis B is of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

10. Let

$$Z = \begin{pmatrix} -8 & 15 \\ -6 & 10 \end{pmatrix}.$$

(a) Find the eigenvalues for Z and for each eigenvalue find an eigenvector.

(b) There is a matrix S so that matrix $S^{-1}ZS$ is a matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a, b \in \mathbb{R}$. What are a, b? What is S?