Math 235 Final Practice Problem Answers

1a. $\left(\begin{matrix}S \text{ is a basis} \\ \text{of } W\end{matrix}\right) \iff$ it spans W and S is independent

1b. P_3 has a basis $\{1, t, t^2, t^3\}$. Thus dimension of P_3 is 4.

The vectors $1, t-1, (t-1)^2, (t-1)^3$ are independent.

Proof $\quad a(1) + b(t-1) + c(t-1)^2 + d(t-1)^3 = 0$

$\Rightarrow d = 0$ since $d(t-1)^3$ is only term with t^3.

$\therefore\ a \cdot 1 + b(t-1) + c(t-1)^2 = 0. \Rightarrow c = 0$ since

$c(t-1)^2$ is only term with t^2 in it. $\Rightarrow$ etc.

$\therefore \{1, t-1, (t-1)^2, (t-1)^3\}$ spans a space of dim 4. Thus this space is all of P_3. $\therefore$ It is a basis

2. $\quad T: P_2 \to P_2$

$f \to f'' - 2f.$

(a) $T: 1 \longrightarrow -2 \cdot 1$

$\updownarrow$ coordinates wrt given basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to -2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\0\\0\end{pmatrix}.$$

(b) $T: x-1 \longmapsto -2(x-1)$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\-2\\0\end{pmatrix}$$

(c) $T: (x-1)^2 \longrightarrow 2 - 2(x-1)^2$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}2\\0\\-2\end{pmatrix}$$

$$\therefore T \longleftrightarrow \begin{pmatrix}-2 & 0 & 2\\ 0 & -2 & 0\\ 0 & 0 & -2\end{pmatrix}.$$

③ (a) False. Let $L : \mathbb{R} \to \mathbb{R}^2$

$t \to t\begin{pmatrix}1\\0\end{pmatrix}$. The equation $L(t) = \begin{pmatrix}0\\1\end{pmatrix}$ does not have a solution.

(b). True. The rank is the dimension of the image. The image is a subspace of $\mathbb{R}^{2010}$. Thus the dim of the image is ≤ 2010.

(c). False. Rank-nullity theorem says $2011 = \text{rank} + \dim \ker$. The rank is ≤ 2010. $\therefore$ dim kernel is ≥ 1.

(d). False. $\left\{ \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} \right\} = \{f_1, f_2\}$ These satisfy $f_1 - f_2 = 0$

This is a non-trivial linear relation.

(e). True. We can obtain reflection about a line L making angle θ with x-axis as composition of

(1) rotate by $-\theta$, (2) reflect about x-axis, (3) rotate by θ.

These are given by matrices ($c = \cos\theta$, $s = \sin\theta$)

(1): $\begin{pmatrix} c & s \\ -s & c \end{pmatrix} = A$ (2) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B$, (3) $\begin{pmatrix} c & -s \\ s & c \end{pmatrix} = D$.

now multiply in order: DBA.

(f). True. $\{v_1 \ldots v_m\} = S$. $\sum a_i v_i = 0 \Rightarrow$

$\langle \sum a_i v_i, v_j \rangle = 0 \Rightarrow a_j \langle v_j, v_j \rangle = a_j = 0$. True for all j.

$\therefore$ all $a_j = 0$. $\therefore$ There is only trivial relation.

(g) True: $L(v + w) = L(v) + L(w)$ (L is linear)

$= 0 + y$.

(h). False. $L(\lambda f) = \lambda f'' - 3\lambda f' + 6\lambda f - 9 \neq \lambda L f =$

$= \lambda(f'' - 3f' + 6\lambda + 9)$.

(i). True. The eigenvalues are roots of the characteristic polynomial which has degree 3. Any poly of degree 3 has a real root, say λ. Let M be ~~an~~ an 3×3 matrix. We have

$\ker(M - \lambda I) \neq \{\vec{0}\}$. ~~Since we~~ Since $\det(M - \lambda I) = 0$.

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4. (a). $T(f+g) = (f+g)' + (f+g) = f' + g' + f + g$

$= f' + f + g' + g = T(f) + T(g)$

$T(\lambda f) = (\lambda f)' + (\lambda f) = \lambda f' + \lambda f = \lambda(f' + f)$

$= \lambda T(f)$

(b) $T: 1 \longmapsto 1$ $\qquad T: t \longmapsto 1+t$ $\qquad T: t^2 \longmapsto 2t + t^2$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}_S \longrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}_S \qquad \begin{pmatrix}0\\1\\0\end{pmatrix}_S \longrightarrow \begin{pmatrix}1\\1\\0\end{pmatrix}_S \qquad \begin{pmatrix}0\\0\\1\end{pmatrix}_S \longrightarrow \begin{pmatrix}0\\2\\1\end{pmatrix}_S$$

$$\therefore T_{SS} = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}.$$

(c). The matrix T_{SS} has 3 leading ones. Thus ker $= 0$ and rank $= 3$. Thus it is invertible. $\therefore$ T_{SS} is an iso and so is T.

(d). We solve. $(T_{SS})\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and translate back to P_2.

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & 1\\ 0 & 1 & 2 & 1\\ 0 & 0 & 1 & 1\end{array}\right) \xrightarrow[\text{OPS}]{\text{ROW}} \left(\begin{array}{ccc|c}1 & 0 & -2 & 0\\ 0 & 1 & 2 & 1\\ 0 & 0 & 1 & 1\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1 & 0 & 0 & +2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\end{array}\right)$$

$\therefore$ $x = +2$ $\quad y = -1$ $\quad z = 1$. This corresponds to $+2 - t + t^2 = p$.

We check: $(+2 - t + t^2) + (-1 + 2t + 0) = 1 + t + t^2$.

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5. (a) $I_{S \leftarrow B} = \begin{pmatrix}1 & 4\\ 1 & 5\end{pmatrix}$ $\therefore$ $I_{B \leftarrow S} = \begin{pmatrix}5 & -4\\ -1 & 1\end{pmatrix}$ $\quad \begin{pmatrix}5 & -4\\ -1 & 1\end{pmatrix}\begin{pmatrix}2\\ -3\end{pmatrix} = \begin{pmatrix}22\\ -5\end{pmatrix}$.

(b) $\begin{pmatrix}5 & -4\\ -1 & 1\end{pmatrix}\begin{pmatrix}9 & -8\\ 10 & -9\end{pmatrix}\begin{pmatrix}1 & 4\\ 1 & 5\end{pmatrix} = \text{LTS}.$

6. Compute the determinant of $B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 5 & 6 & 0 \\ 0 & 7 & 8 & 0 \\ 3 & 0 & 0 & 4 \end{pmatrix}$

Use minors: (FIRST ROW)

$$\det B = 1\det\begin{pmatrix} 5&6&0\\7&8&0\\0&0&4\end{pmatrix} - 2\det\begin{pmatrix} 0&5&6\\0&7&8\\3&0&0\end{pmatrix} = 1\cdot(40-42)(4) - 2\cdot 3(40-42)$$

$$= 4(-2) - 6(-2) = -8+12 = 4$$

$$\det B^{235} = 4^{235}. \qquad \det B^{-1} = \frac{1}{4}.$$

7. (a) $\chi_C(\lambda)$ = characteristic poly of $C = \det(C-\lambda I) = \det\begin{pmatrix}4-\lambda & -1\\ 1 & 2-\lambda\end{pmatrix}$

$$= \lambda^2 - 6\lambda + 9 = (\lambda-3)^2.$$

(b). $\begin{pmatrix} 4-3 & -1 \\ 1 & 2-3\end{pmatrix} = \begin{pmatrix}1&-1\\1&-1\end{pmatrix} \xrightarrow[\text{Row ops}]{} \begin{pmatrix}1&-1\\0&0\end{pmatrix}$

$\Rightarrow y\begin{pmatrix}1\\1\end{pmatrix}$ are all eigenvectors ($y \neq 0$) for eigenvalue 3.

(c). No. The eigenvectors fail to span $\mathbb{R}^2$.

8. (a) $\chi_C(\lambda) = \det\begin{pmatrix}-7-\lambda & 3\\ -18 & 8-\lambda\end{pmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$

(b) $\lambda = 2$, $\lambda = -1$ are the eigenvectors for C.
For $\lambda = 2$ we find an eigenvector.

$$(C-\lambda I) = C - 2I = \begin{pmatrix}-7-2 & 3\\ -18 & 8-2\end{pmatrix} = \begin{pmatrix}-9&3\\-18&6\end{pmatrix}$$

$\rightarrow$ Row operations $\begin{pmatrix}-9&3\\0&0\end{pmatrix} \rightarrow \begin{pmatrix}-3&1\\0&0\end{pmatrix}$ $\therefore y\begin{pmatrix}1\\3\end{pmatrix}$

for $y \neq 0$ are eigenvectors for eigenvalue $\lambda = 2$.

For $\lambda = -1$. $C - \lambda I = \begin{pmatrix}-6&3\\-18&9\end{pmatrix} \xrightarrow[\text{Row ops}]{} \begin{pmatrix}-2&1\\0&0\end{pmatrix}$

$\therefore \begin{pmatrix}1\\2\end{pmatrix}$ is eigenvector — (as well as any mult by $\neq 0$ scalars).
for eigenvector -1.

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8c. Yes — Let $\mathcal{B}$ = basis $\left\{\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$. Then WRT this basis our linear map is

$$D = \begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix} = \underbrace{-\begin{pmatrix}2 & -1\\ -3 & 1\end{pmatrix}}_{B^{-1}} \underbrace{\begin{pmatrix}-7 & 3\\ -18 & 8\end{pmatrix}}_{C} \underbrace{\begin{pmatrix}1 & 1\\ 3 & 2\end{pmatrix}}_{B}$$

9. Let $A = \begin{pmatrix}1/3 & 10/3\\ -4/3 & 5/3\end{pmatrix}$. The characteristic polynomial of A is

$$\det\begin{pmatrix}1/3-\lambda & 10/3\\ -4/3 & 5/3-\lambda\end{pmatrix} = \lambda^2 - 2\lambda + 5.$$ The eigen values of A

~~are~~ $\cancel{\dfrac{2 \pm \sqrt{4-4\cdot 9}}{2} = \dfrac{2\pm\sqrt{-32}}{2} = 2 \pm \ldots}$

are $\dfrac{2 \pm \sqrt{4-4\cdot 5}}{2} = \dfrac{2\pm\sqrt{-16}}{2} = 1 \pm 2i.$

We find an eigen vector for the eigen value $1+2i$.

$$\begin{pmatrix}1/3-(1+2i) & 10/3\\ -4/3 & 5/3-(1+2i)\end{pmatrix} = \begin{pmatrix}-2/3-2i & \frac{10}{3}\\ -4/3 & 2/3-2i\end{pmatrix}$$

Row ops. $\begin{pmatrix}-2/3-2i & 10/3\\ 0 & 0\end{pmatrix} \Rightarrow (-2/3-2i)x + \frac{10}{3}y = 0$

set $x=1$ you get $y = \frac{3}{10}(+2/3+2i) = +1/5 + \frac{6}{10}i$

$= -1/5 - \frac{3}{5}i$

$\therefore \begin{pmatrix}1\\ +1/5+3/5\,i\end{pmatrix}$ is an eigenvector for $1+2i$

$= \begin{pmatrix}1\\ +1/5\end{pmatrix} + i\begin{pmatrix}0\\ 3/5\end{pmatrix} = u + iv$ notation.

Let $B = \{v, u\}$. The matrix of this linear map WRT $\{v,u\}$ is $\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$.

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10. The characteristic polynomial of $\begin{pmatrix} -8 & 15 \\ -6 & 10 \end{pmatrix}$ is $\lambda^2 - 2\lambda + 10$.

The roots of this are $\dfrac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$. These are the eigenvalues. We find an eigenvector for the eigenvalue $\lambda = 1+3i$. It is any $\neq 0$ vector in the nullspace of $\begin{pmatrix} -8-(1+3i) & 15 \\ -6 & 10-(1+3i) \end{pmatrix} = \begin{pmatrix} -9-3i & 15 \\ -6 & 9-3i \end{pmatrix}$.

The eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ is any $\neq 0$ solution to $(-9-3i)x + 15y = 0$

Set $x=1$ (rather arbitrarily). get $15y = 9+3i \Rightarrow y = 9/_{15} + 3/_{15}\, i$

$y = 3/_5 + 1/_5\, i$. The vector is $\begin{pmatrix} 1 \\ 3/_5 + 1/_5\, i \end{pmatrix} = \begin{pmatrix} 1 \\ 3/_5 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1/_5 \end{pmatrix} = u + iv$.

We chose a basis $(v, u) = \left\{ \begin{pmatrix} 0 \\ 1/_5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3/_5 \end{pmatrix} \right\}$. Let $S = \begin{pmatrix} 0 & 1 \\ 1/5 & 3/5 \end{pmatrix}$.

Then
$$S^{-1} Z S = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}.$$

An eigenvector for eigenvalue $1-3i$ is $u - iv$.