DEPARTMENT OF MATHEMATICS AND STATISTICS
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MATH 235 **MIDTERM** **Fall 2010**

1: True or False. (Please support your answer with a brief reason or a counter-example.)

1a: Let M be an $n \times n$ matrix. If the columns of M are independent, then the kernel of M is just the zero vector.

1b: If the set of vectors $\{u, v, w\}$ is independent, then w must be a linear combination of u and v.

1c: The image of a 3×4 matrix M is a subspace of $\mathbb{R}^3$.

1d: The function

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$$

is linear.

1e: The set of vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ so that $x^2 + y^2 + z^2 = 1$ is a subspace of $\mathbb{R}^3$.

2: Consider the following system of equations:

$$\begin{aligned} -x - z - 2w &= 0 \\ 2x + y + 3z + 5w &= 3 \\ -x + y - w &= 3 \end{aligned}$$

2a: Express this as a matrix equation $AX = B$ with
$A =$ and $B =$

2b: Find all solutions $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ using row reduction (Gaussian elimination).

3: Find a system of two linear equation that a, b, c, d must satisfy so that

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4$$

is in the span of the set of vectors

$$S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

4a: Let $F : \mathbb{R}^n \to \mathbb{R}^m$. Define what it means for F to be linear.

4b: Suppose a linear map $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is given by the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & -1 \\ 1 & -2 & -1 & -3 \\ 2 & 1 & 3 & 1 \end{pmatrix}.$$

Find a basis for $\text{im}(F)$ and compute the rank of F.
4c: Let F be the map in 4b. Find a basis for $\ker(F)$ and compute its dimension (the nullity of F).
5: The image of a matrix M of size 5×5 has dimension 2.
5a: How many independent columns vectors does M have?
5b: What is the dimension of the kernel of M?
5c: Does the equation $MX = b$ have a solution for every $b \in \mathbb{R}^5$? Why?

6a: Find a basis for the subspace $V \subset \mathbb{R}^4$ of all vectors orthogonal to $u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \end{pmatrix}$.
6b: What is the dimension of this subspace V in part 6a?
7: Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map. Let

$$u = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad v = \begin{pmatrix} -2 \\ 4 \\ 0 \\ 4 \end{pmatrix}.$$

Assume that

$$f(u) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and that $f(v) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. What is $f(u + v)$? Explain why.