DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
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EXAM 2

(1) (18 points) You are given below the matrix A together with its row reduced echelon form B

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 1 & 4 \\ 2 & 0 & 4 & 3 & 5 & 2 \\ 3 & 2 & 4 & 6 & 9 & 10 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

You do **not** need to check that A and B are indeed row equivalent.

(a) Find a basis for the kernel $\ker(A)$ of A.

Solution: The vectors

$$\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix},$$

are linearly independent vectors in the kernel of A. Since the nullity of A is 3, the vectors form a basis for the kernel of A.

(b) Find a basis for the image $im(A)$ of A.

Solution: The vectors

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 6 \end{pmatrix},$$

form a basis for the image of A since the 3rd, 5th and 6th columns of A are redundant vectors among the columns of A.

(c) Does the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ belong to the image of A? Use part 1b to minimize your computations. Justify your answer!

Solution: Yes, since

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \\ 6 \end{pmatrix}.$$

(2) (12 points)

(a) Let $T : \mathbb{R}^7 \to \mathbb{R}^4$ be a linear transformation. What are the possible values of $\dim(\ker(T))$? Justify your answer!

Answer: The equality $\dim(\ker(T)) + \dim(im(T)) = 7$ holds, by the Rank-Nullity Theorem. The inequality $\dim(im(T)) \leq 4$ holds, since $im(T)$ is a subspace of $\mathbb{R}^4$. Hence, $\dim(\ker(T)) = 7 - \dim(im(T)) \geq 7 - 4 = 3$. The inequality $\dim(\ker(T)) \leq 7$ holds, since $\ker(T)$ is a subspace of $\mathbb{R}^7$. We conclude that $3 \leq \dim(\ker(T)) \leq 7$.

(b) Let A and B be $n \times n$ matrices. Assume that $AB = 0$. Show that the image of B is contained in the kernel of A.

Answer: Let $\vec{y}$ be a vector in the image of B. Then $\vec{y} = B\vec{x}$, for some $\vec{x}$ in $\mathbb{R}^n$, by definition of the image of B. The vector $\vec{y}$ is in the kernel of A, if $A\vec{y} = \vec{0}$. The latter is indeed the case, since we have

$$A\vec{y} = A(B\vec{x}) = (AB)\vec{x} = \vec{0},$$

where the rightmost equality follows from the assumption that $AB = 0$.

(c) Let A and B be $n \times n$ matrices and assume that the image of B is contained in the kernel of A. Show that $\text{rank}(B) \leq \dim(\ker(A))$. Explain why it follows that $\text{rank}(A) + \text{rank}(B) \leq n$.

Answer: The equality $\text{rank}(B) = \dim(im(B))$ is the definition of $\text{rank}(B)$. The inequality $\dim(im(B)) \leq \dim(\ker(A))$ holds, since $im(B)$ is assumed a subspace of $\ker(A)$. We conclude the inequality $\text{rank}(A) + \text{rank}(B) \leq \text{rank}(A) + \dim(\ker(A))$. Now the right hand side is n, by the Rank-Nullity Theorem. We conclude the inequality $\text{rank}(A) + \text{rank}(B) \leq n$.

(3) (18 points) Let $\vec{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(a) Show that $\{\vec{v}_2, \vec{v}_3\}$ form a basis for the subspace P of $\mathbb{R}^3$ orthogonal to $\vec{v}_1$.

Answer: We have that $P = \ker \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Since P is the kernel of a 1×3 matrix of rank 1, it is a vector space (Theorem 3.2.2) of dimension $\dim P = 3 - 1 = 2$ (by the rank-nullity theorem). Geometrically, P is a plane through the origin in $\mathbb{R}^3$ with normal vector $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We have

$$v_1 \cdot v_2 = \frac{1}{\sqrt{3}}(1 - 1 + 0) = v_1 \cdot v_3 = 0.$$

Hence $v_2 \in P$ and $v_3 \in P$. They are linearly independent, since the 3-rd entry of v_2 is 0 and the 3-rd entry of v_3 is $-1 \neq 0$. (Theorem 3.2.5, p.117). Since $\dim P = 2$, $\text{span}\{v_2, v_3\} = P$.

(b) Consider the basis $\beta := \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(\vec{x}) = \vec{x} - 2(\vec{v}_1 \cdot \vec{x})\vec{v}_1$. Find the β-matrix B of T (the matrix of T in the basis β). Justify your answer!

Answer: By Theorem 4.3.2, p.174 (or Definition 3.4.3, p.143) we have

$$B = \begin{bmatrix} T(v_1)_\beta & T(v_2)_\beta & T(v_3)_\beta \end{bmatrix}.$$

By part a), $P = \text{span}\{v_2, v_3\}$ and v_3 is orthogonal to P. The linear transformation T is a reflection with respect to the plane P, hence

$$T(v_1) = -v_1,\ T(v_2) = v_2,\ T(v_3) = v_3,$$

and thus

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Let S be the 3×3 matrix $(\vec{v}_1 \vec{v}_2 \vec{v}_3)$ with columns $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$. Express the standard matrix A of T in terms of the matrix S and the matrix B you found in part 3b. (You do not need to simplify your answer).

Answer: We have the following commutative diagramme (pp.145, 174)

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{A} & \mathbb{R}^3 \\ {\scriptstyle S^{-1}=L_\beta}\downarrow & & \downarrow{\scriptstyle S^{-1}=L_\beta} \\ \mathbb{R}^3 & \xrightarrow{B} & \mathbb{R}^3 \end{array},$$

(see Definition 4.1.3 for L_β.) Thus

$$A = SBS^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 1 & 0 \\ \frac{1}{\sqrt{3}} & -1 & 1 \\ \frac{1}{\sqrt{3}} & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 1 & 0 \\ \frac{1}{\sqrt{3}} & -1 & 1 \\ \frac{1}{\sqrt{3}} & 0 & -1 \end{bmatrix}^{-1}.$$

(4) (18 points) Let $\mathbb{R}^{2\times 2}$ be the vector space of 2×2 matrices and P an invertible 2×2 matrix. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the function sending a matrix M to $T(M) = P^{-1}MP$.

(a) Show that T is a linear transformation.

Sketch of answer: One shows that $T(M+N) = T(M) + T(N)$ and $T(kM) = kT(M)$.

(b) Show that T is an isomorphism by explicitly finding T^{-1}. Carefully justify your answer!

Sketch of answer: We have $T^{-1}(M) = PMP^{-1}$ because $T(T^{-1}(M)) = M$.

(c) Assume now that $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Find the matrix B of T in part 4a in the basis $\beta := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $\mathbb{R}^{2\times 2}$.

Sketch of answer: First find $P^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ to compute $T(e_1) = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$, and $T(e_4) = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$. Then the matrix is

$$B = \begin{pmatrix} 2 & 1 & -2 & -1 \\ 1 & 1 & -1 & -1 \\ -2 & -1 & 4 & 2 \\ -1 & -1 & 2 & 2 \end{pmatrix}.$$

(5) (16 points) Let P_2 be the vector space of polynomials of degree ≤ 2. (i) Which of the following subsets W of P_2 are subspaces? In each case verify the three conditions in the definition of a subspace, or demonstrate that one of them is violated.
(ii) Find a basis for those that are subspaces.

(a) $W = \{f(t) \ : \ f'(0) = 1\}$ is the subset of polynomial functions $f(t)$, such that the value of its derivative at $t = 0$ is 1.

(b) $W = \{f(t) \ : \ f(1) = f'(2)\}$.

Answer:

(a) i) W is not a subspace, since it does not contain the zero polynomial. (If $f(t) = 0$, then $f'(1) = 0 \neq 1$).
ii) not applicable.

(b) i) W is a subspace because
  (i) It contains the zero polynomial (If $f(t) = 0$, then $f(1) = f'(2) = 0$).
  (ii) It's closed under addition: If $f(t)$ and $g(t)$ are in W, then

$$(f+g)(1) = f(1) + g(1) = f'(2) + g'(2) = (f+g)'(2).$$

  (iii) It's closed under scalar multiplication: If $f(t)$ is in W and k is a scalar, then

$$(kf)(1) = k \cdot f(1) = k \cdot f'(2) = (kf)'(2).$$

ii) If $f(t) = a + bt + ct^2$ then $f(1) = a + b + c$ and $f'(2) = b + 4c$, so f is in W if and only if

$$a + b + c = b + 4c.$$

We see b can be anything and $a = 3c$. So a general element of W is of the form $3c + bt + ct^2 = b(t) + c(3 + t^2)$ and a basis of W is $\{t, 3 + t^2\}$.

(6) (18 points) Let $T : P_2 \to \mathbb{R}^3$ be the linear transformation given by $T(f(t)) = \begin{bmatrix} f'(0) \\ f(1) \\ f(-1) \end{bmatrix}$.

The first entry on the right hand side above is the value of the *derivative* f' at 0.

(a) Find a basis (consisting of *polynomials*) for the kernel $\ker(T)$. Carefully justify why the set you found is a basis.

Solution: Suppose $f(t) = a + bt + ct^2$ is in $\ker(T)$. Then $T(f(t)) = 0$, i.e. $\begin{bmatrix} f'(0) \\ f(1) \\ f(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. In other words $\begin{bmatrix} b \\ a+b+c \\ a-b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving these three equations, we see that

$b = 0$ and $a = -c$, so f is of the form $f(t) = -c + ct^2 = c(-1 + t^2)$. This shows that every polynomial in $\ker(T)$ is a scalar multiple of $-1 + t^2$, so $\{-1 + t^2\}$ is a basis for $\ker(T)$.

(b) Use your answer in part 6a in order to determine the rank and nullity of T. Justify your answer!

Solution: From part (a), $\text{nullity}(T) = \dim(\ker(T)) = 1$. By the rank-nullity theorem, $\dim(P_2) = \text{rank}(T) + \text{nullity}(T)$, i.e. $3 = \text{rank}(T) + 1$, so $\text{rank}(T) = 2$.

(c) Find a basis for the image $im(T)$. Justify your answer!

Solution: If $\vec{v}$ is in $\text{im}(T)$, then $\vec{v} = T(a + bt + ct^2)$ for some polynomial $a + bt + ct^2$. This means

$$\vec{v} = T(a + bt + ct^2) = \begin{bmatrix} b \\ a + b + c \\ a - b + c \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + (a + c) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

So every element of the image of T is a linear combination of the linearly independent vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, so a basis for $\text{im}(T)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.