MATH 235 SPRING 2011
EXAM 1 SOLUTIONS

(1) (16 points)

a) Show that the **reduced** row echelon form of the augmented matrix of the system

$$\begin{aligned} x_1 + x_2 + 2x_4 + x_5 &= 3 \\ x_1 - x_3 + x_4 + x_5 &= 2 \\ -2x_1 + 2x_3 - 2x_4 - x_5 &= -3 \end{aligned}$$

is $\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Use at most six elementary row operations. (Partial credit will be given if you use more). Clearly write in words each elementary row operation you use.

Solution:

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 1 & 3 \\ 1 & 0 & -1 & 1 & 1 & 2 \\ -2 & 0 & 2 & -2 & -1 & -3 \end{bmatrix}$$

1. Subtract row 1 from row 2
2. Add twice row 1 to row 3

$$\longrightarrow \begin{bmatrix} 1 & 1 & 0 & 2 & 1 & 3 \\ 0 & -1 & -1 & -1 & 0 & -1 \\ 0 & 2 & 2 & 2 & 1 & 3 \end{bmatrix}$$

3. Multiply row 2 by (-1)

$$\longrightarrow \begin{bmatrix} 1 & 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 2 & 1 & 3 \end{bmatrix}$$

4. Subtract row 2 from row 1
5. Subtract twice row 2 from row 3

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

6. Subtract row 3 from row 1

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

b) Find the general solution of the system.

Solution: From the reduced row echelon form of the system, we can see that x_3 and x_4 are the free variables. Letting $x_3 = s$ and $x_4 = t$, we have the general solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} s - t + 1 \\ -s - t + 1 \\ s \\ t \\ 1 \end{bmatrix}$$

for all $s, t \in \mathbb{R}$.

(2) (16 points) Let A be a 5×3 matrix (5 rows and 3 columns), $\vec{b}$, $\vec{c}$, $\vec{d}$ three vectors in $\mathbb{R}^5$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, with variables x_1, x_2, x_3. You are told that the matrix equation $A\vec{x} = \vec{b}$ has a unique solution. Carefully justify using complete sentences your answers to the following questions.

(a) What is the row reduced echelon form of A?

Solution: Since the matrix equation $A\vec{x} = \vec{b}$ has a unique solution, the matrix A must have rank equal to the number of columns. Thus

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In particular, since the rank of A is 3, there is *at most* one solution to any system of the form $A\vec{x} = \vec{v}$.

(b) What can you say about the number of solutions of the system $A\vec{x} = \vec{0}$?

Solution: The system is consistent, and has the unique solution given by the zero vector $\vec{0}$ in $\mathbb{R}^3$.

(c) You are given the additional information that the system $A\vec{x} = \vec{c}$ is consistent. What can you say about the number of solutions of the system $A\vec{x} = \vec{b} + \vec{c}$.

Solution: Suppose that $\vec{x}_b$ and $\vec{x}_c$ are the unique solutions to the systems $A\vec{x} = \vec{b}$ and that $A\vec{x} = \vec{c}$ respectively. Then

$$A(\vec{x}_b + \vec{x}_c) = A\vec{x}_b + A\vec{x}_c = \vec{b} + \vec{c}$$

and so there is one unique solution to the system $A\vec{x} = \vec{b} + \vec{c}$.

(d) What can you say about the number of solutions of the system $A\vec{x} = \vec{d}$?

Solution: Since the rank of A is 3, there is at most one solution to the problem.

(3) (18 points) You can solve parts b and c below even without solving part a.

a) Let L be the line in $\mathbb{R}^2$ through the origin and $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Recall that the reflection $Ref_L : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by the formula

$$Ref_L(\vec{x}) = \frac{2(\vec{x}\cdot\vec{v})}{(\vec{v}\cdot\vec{v})}\vec{v} - \vec{x}, \tag{1}$$

where $\vec{x}\cdot\vec{v}$ is the dot product of $\vec{x}$ and $\vec{v}$. Use the above formula to find the matrix A of Ref_L, so that $Ref_L(\vec{x}) = A\vec{x}$, for all vectors $\vec{x}$ in $\mathbb{R}^2$. Credit will not be given for an answer which does not derive the entries of A from equation (1) above.

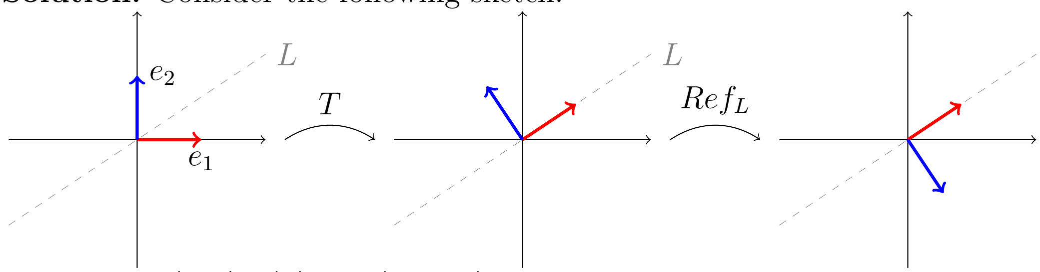
Solution: We compute

$$\begin{aligned} Ref_L(\vec{x}) &= \frac{2(\vec{x}\cdot\vec{v})}{(\vec{v}\cdot\vec{v})}\vec{v} - \vec{x} \\ &= \frac{2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 2 \end{bmatrix}}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 2 \end{bmatrix}}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \frac{2(3x_1 + 2x_2)}{9+4}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \frac{6x_1 + 4x_2}{13}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \frac{1}{13}\begin{bmatrix} 18x_1 + 12x_2 \\ 12x_1 + 8x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \frac{1}{13}\begin{bmatrix} 5x_1 + 12x_2 \\ 12x_1 - 5x_2 \end{bmatrix} \\ &= \frac{1}{13}\begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}\vec{x}. \end{aligned}$$

Thus $A = \frac{1}{13}\begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$.

b) Let θ be the angle from the x_1-axis in $\mathbb{R}^2$ to the line L in part a. Denote by $T : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation of the plane an angle θ counterclockwise about the origin. Note that T maps the x_1-axis onto L and the x_2-axis onto the line perpendicular to L. Use geometric considerations, justified via both sketches and complete sentences, in order to compute the following:

Solution: Consider the following sketch:



i) $Ref_L\left(T\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

Solution: The transformation T sends the vector $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ to the vector $T(e_1) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$. Since $T(e_1)$ is parallel to line L, its reflection across L is $T(e_1)$.

ii) $Ref_L\left(T\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}$

Solution: The transformation T sends the vector $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$ to the vector $T(e_2) = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$. Since $T(e_2)$ is perpendicular to L, its reflection across L is $-T(e_2)$.

c) Let B be the matrix of T in part b. Use your work in part b to prove the equality $AB = \frac{1}{\sqrt{13}}\begin{pmatrix}3 & 2\\ 2 & -3\end{pmatrix}$.

Solution: We know that the first column of the matrix $AB = Ref_L(T(e_1))$ and that the second column of $AB = Ref_L(T(e_2))$. Since θ is the angle between the x_1-axis and the line L, we have $\cos\theta = \frac{3}{\sqrt{13}}$ and $\sin\theta = \frac{2}{\sqrt{13}}$, which verifies the equality.

(4) (16 points) Find all matrices $M = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ that commute with the matrix $A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$, i.e., which satisfy

$$AM = MA. \tag{2}$$

Follow the following three steps.

a) Translate the equation (2) to a system of linear equations that the variables w, x, y, and z should satisfy, in order for M and A to commute.

Solution: We write out explicitly the equation $AM = MA$:

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \iff$$

$$\begin{bmatrix} 2y & 2z \\ w+3y & x+3z \end{bmatrix} = \begin{bmatrix} x & 2w+3x \\ z & 2y+3z \end{bmatrix} \iff$$

$$\left| \begin{array}{l} 2y = x \\ 2z = 2w + 3x \\ w + 3y = z \\ x + 3z = 2y + 3z \end{array} \right. \Leftrightarrow \left| \begin{array}{l} 2y = x \\ 2z = 2w + 3x \\ w + 3y = z \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x - 2y = 0 \\ 2w + 3x - 2z = 0 \\ w + 3y - z = 0 \end{array} \right.$$

Here we discarded the fourth equation: after cancelling $3z$ it reduces to the first one.

b) Find the general solution $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$ of the system in part a.

Solution: At this point you can follow one of two routes.
Method 1: Gauss Elimination
The system we want to solve is homogeneous and has coefficient matrix

$$\begin{bmatrix} 0 & 1 & -2 & 0 \\ 2 & 3 & 0 & -2 \\ 1 & 0 & 3 & -1 \end{bmatrix}$$

Swapping the first and third rows we obtain

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 2 & 3 & 0 & -2 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 3 & -6 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here the first operation is $R_2 \mapsto R_2 - 2R_1$. For the second step we observe that R_2 and R_3 are multiples (if you prefer, $R_2 \mapsto \frac{1}{3}R_2$, $R_3 \mapsto R_3 - R_2$). Then the general solution is

$$\left| \begin{array}{l} w = t - 3s \\ x = 2s \\ y = s \\ z = t \end{array} \right. , s, t \in \mathbb{R}.$$

$$\Leftrightarrow \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R}$$

Method 2

You can solve the system

$$\left| \begin{array}{l} x - 2y = 0 \\ 2w + 3x - 2z = 0 \\ w + 3y - z = 0 \end{array} \right.$$

"by hand". First we observe that the second equation can be eliminated: if $x = 2y$ and $w = z - 3y$, then $2w + 3x - z = 6z - 6y + 6y - 2z = 0$. Thus we have to solve

$$\left| \begin{array}{l} x - 2y = 0 \\ w + 3y - z = 0 \end{array} \right. .$$

Note: We did the same two operations as in the row-reduction above!

There are many (five) ways to choose two of the variables as independent (Gauss elimination would have made this choice for you!). One popular choice was to take w and y as parameters, so $x = 2y$ and $z = w + 3y$, that is:

$$\left| \begin{array}{l} w = t \\ x = 2r \\ y = r \\ z = t + 3r \end{array} \right. , r, t \in \mathbb{R},$$

that is,

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, t, r \in \mathbb{R}.$$

Note: **Method 1** and **Method 2** give the same set of solutions!

c) Find the general form of a matrix M, which commutes with A.

Solution:

Here we just copy our result in matrix form. That is,

$$M = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} t - 3s & 2s \\ s & t \end{bmatrix} =$$

$$= s \begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, s, t \in \mathbb{R}.$$

If you did not use Gauss elimination but followed **Method 2**, you result would look like

$$M = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} t & 2r \\ r & t + 3r \end{bmatrix} =$$

$$= t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = tI_2 + rA, \ t, r \in \mathbb{R}.$$

Observe:

Our general solution is a linear combination of two natural candidates: the identity matrix I_2 and the matrix A itself! It is clear that both I_2 and A commute with A, and so does any linear combination of theirs. Interestingly, this gives us *all* matrices commuting with A. But then, since $A^2 = AA$ commutes with A (why?), A^2 must be a linear combination of I_2 and A. Consequently, any power A^n, $n \geq 2$ is a combination of I_2 and A.

(5) (a) (7 points) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. Compute A^{-1}. Show all your work.

Answer: Row reduce $(A|I) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{pmatrix} \sim$ $\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}$. Hence, $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$.

(b) (9 points) Determine which of the following linear transformations T from $\mathbb{R}^2$ to $\mathbb{R}^2$ are invertible. Give a reason, if it is not invertible. If the inverse exists describe it geometrically.

(i) T is the rotation of $\mathbb{R}^2$ 45 degrees counterclockwise.

Answer: T is invertible. Its inverse is the rotation of the plane 45 degrees clockwise.

(ii) T is the reflection of $\mathbb{R}^2$ with respect to a line L through the origin and a non-zero vector $u = (u_1, u_2)$.

Answer: T is invertible. T is its own inverse, $T^{-1} = T$.

(iii) T is the projection of $\mathbb{R}^2$ onto the line L in part 5(b)ii.

Answer: T is **not** invertible. A function $T : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible, if the equation $T(\vec{x}) = \vec{y}$ has a unique solution $\vec{x}$, for every $\vec{y} \in \mathbb{R}^2$. If $\vec{y}$ does not belong to the line L, the equation does not have any solution. If $\vec{y}$ belongs to L, the equation has infinitely many solutions (all vectors on the line through $\vec{y}$ orthogonal to L).

(6) (18 points)

a) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation $T(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 2 & 2 \\ 1 & 2 & 1 & 7 & 2 \end{pmatrix}$.

You are given that A is row equivalent to the matrix $B = \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

You do **not** need to verify this fact. Find a basis for the kernel of T. In other words, find a set of vectors which span $\ker(T)$ and which is linearly independent. **Explain** why the set you found spans $\ker(T)$ and why it is linearly independent.

Solution: Let $(x_1, x_2, x_3, x_4, x_5)$ be the coordinates for $\mathbb{R}^5$. Then by just solving the corresponding linear system equal to zero, one gets as general solution $x_2(-2, 1, 0, 0, 0) + x_4(-3, 0, -4, 1, 0)$. So, these two vectors span the kernel. They are linearly independent because clearly one is not scalar multiple of the other.

b) Let L be the line in $\mathbb{R}^3$ spanned by the vector $\vec{v} := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Denote by $L^\perp$ the set of all vectors $\vec{x}$ in $\mathbb{R}^3$ that are orthogonal to L (i.e., to $\vec{v}$). So $L^\perp$ consists of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, such that the dot product $\vec{v} \cdot \vec{x} = 0$ is zero. Show that $L^\perp$ is a *subspace* of $\mathbb{R}^3$ by stating the three properties defining a subspace and verifying that $L^\perp$ satisfies each of them.

Solution: One checks the three properties. Let $\vec{x}, \vec{y}$ be orthogonal to $\vec{v}$ and let t be a number, then (i) zero vector is there because $\vec{0} \cdot \vec{v} = 0$. (ii) the sum is there because $\vec{v} \cdot (\vec{x} + \vec{y}) = \vec{v} \cdot \vec{x} + \vec{v} \cdot \vec{y} = 0 + 0 = 0$. (iii) scalar multiplication is there because $\vec{v} \cdot (t\vec{x}) = t(\vec{v} \cdot \vec{x}) = 0$.