DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MATH 235 SPRING 2011
EXAM 1

1. (16 points) a) Show that the **reduced** row echelon form of the augmented matrix of the system

$$\begin{aligned} x_1 + x_2 + 2x_4 + x_5 &= 3 \\ x_1 - x_3 + x_4 + x_5 &= 2 \\ -2x_1 + 2x_3 - 2x_4 - x_5 &= -3 \end{aligned}$$

is $\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Use at most six elementary row operations. (Partial credit will be given if you use more). Clearly write in words each elementary row operation you use.

b) Find the general solution of the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} =$$

2. (16 points) Let A be a 5×3 matrix (5 rows and 3 columns), $\vec{b}$, $\vec{c}$, $\vec{d}$ three vectors in $\mathbb{R}^5$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, with variables x_1, x_2, x_3. You are told that the matrix equation $A\vec{x} = \vec{b}$ has a unique solution. Carefully justify using complete sentences your answers to the following questions.

   (a) What is the row reduced echelon form of A?

   (b) What can you say about the number of solutions of the system $A\vec{x} = \vec{0}$?

   (c) You are given the additional information that the system $A\vec{x} = \vec{c}$ is consistent. What can you say about the number of solutions of the system $A\vec{x} = \vec{b} + \vec{c}$.

   (d) What can you say about the number of solutions of the system $A\vec{x} = \vec{d}$?

3. (18 points) You can solve parts b and c below even without solving part a.

a) Let L be the line in $\mathbb{R}^2$ through the origin and $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Recall that the reflection $Ref_L : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by the formula

$$Ref_L(\vec{x}) = \frac{2(\vec{x} \cdot \vec{v})}{(\vec{v} \cdot \vec{v})}\vec{v} - \vec{x}, \tag{1}$$

where $\vec{x} \cdot \vec{v}$ is the dot product of $\vec{x}$ and $\vec{v}$. Use the above formula to find the matrix A of Ref_L, so that $Ref_L(\vec{x}) = A\vec{x}$, for all vectors $\vec{x}$ in $\mathbb{R}^2$. Credit will not be given for an answer which does not derive the entries of A from equation (1) above.

b) Let θ be the angle from the x_1-axis in $\mathbb{R}^2$ to the line L in part a. Denote by $T : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation of the plane an angle θ counterclockwise about the origin. Note that T maps the x_1-axis onto L and the x_2-axis onto the line perpendicular to L. Use geometric considerations, justified via both sketches and complete sentences, in order to compute the following:

i) $Ref_L\left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) =$

ii) $Ref_L\left(T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) =$

c) Let B be the matrix of T in part b. Use your work in part b to prove the equality $AB = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}$. Hint: Avoid computing B, rather compute AB directly.

4. (16 points) Find all matrices $M = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ that commute with the matrix $A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$, i.e., which satisfy

$$AM = MA. \tag{2}$$

Follow the following three steps.
a) Translate the equation (2) to a system of linear equations that the variables w, x, y, and z should satisfy, in order for M and A to commute.

b) Find the general solution $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$ of the system in part a.

c) Find the general form of a matrix M, which commutes with A.

5. (a) (7 points) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. Compute A^{-1}. Show all your work.

   (b) (9 points) Determine which of the following linear transformations T from $\mathbb{R}^2$ to $\mathbb{R}^2$ are invertible. Give a reason, if it is not invertible. If the inverse exists describe it geometrically.

   i. T is the rotation of $\mathbb{R}^2$ 45 degrees counterclockwise.

   ii. T is the reflection of $\mathbb{R}^2$ with respect to a line L through the origin and a non-zero vector $u = (u_1, u_2)$.

iii. T is the projection of $\mathbb{R}^2$ onto the line L in part 5(b)ii.

6. (18 points) a) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation $T(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 2 & 2 \\ 1 & 2 & 1 & 7 & 2 \end{pmatrix}$. You are given that A is row equivalent to the matrix $B = \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. You do **not** need to verify this fact. Find a basis for the kernel of T. In other words, find a set of vectors which span $\ker(T)$ and which is linearly independent. **Explain** why the set you found spans $\ker(T)$ and why it is linearly independent.

   b) Let L be the line in $\mathbb{R}^3$ spanned by the vector $\vec{v} := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Denote by $L^\perp$ the set of all vectors $\vec{x}$ in $\mathbb{R}^3$ that are orthogonal to L (i.e., to $\vec{v}$). So $L^\perp$ consists of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, such that the dot product $\vec{v} \cdot \vec{x} = 0$ is zero. Show that $L^\perp$ is a *subspace* of $\mathbb{R}^3$ by stating the three properties defining a subspace and verifying that $L^\perp$ satisfies each of them.