Math 235 section 4 Midterm 1 Spring 2009

Name: My Solution

(9 pts)

1. (15 points) a) Show that the row **reduced** echelon form of the augmented matrix of the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + 3x_5 &= 1 \\ 2x_1 + x_2 + x_4 + 4x_5 &= 1 \\ x_1 - x_3 + x_4 + 2x_5 &= 0 \end{aligned}$$

is $\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$. Use at most seven elementary operations. Show all your work. Clearly write in words each elementary row operation you used.

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 3 & 1 \\ 2 & 1 & 0 & 1 & 4 & 1 \\ 1 & 0 & -1 & 1 & 2 & 0 \end{array}\right) \underset{\text{Add } -R_1 \text{ to } R_3}{\overset{\text{Add } -2R_1 \text{ to } R_2}{\sim}} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 3 & 1 \\ 0 & -1 & -2 & -1 & -2 & -1 \\ 0 & -1 & -2 & 0 & -1 & -1 \end{array}\right) \underset{\text{Multiply } R_2 \text{ by } -1}{\overset{\text{Add } -R_2 \text{ to } R_3}{\sim}}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right) \underset{\text{Add } -R_3 \text{ to } R_1}{\overset{\text{Add } -R_3 \text{ to } R_2}{\sim}} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right) \underset{\text{Add } -R_2 \text{ to } R_1}{\sim}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

x_3 x_5 are free

6 pts

b) Find the general solution for the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_3 - x_5 \\ -2x_3 - x_5 + 1 \\ x_3 \\ -x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2. a) (8 points) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\underbrace{\begin{matrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{matrix}}_{= A^{-1}}$$

b) (2 points) Use matrix multiplication to check that the matrix you found is indeed A^{-1}.

$$AA^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}\begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) (5 points) Let A, B, C be $n \times n$ matrices, with A and B invertible, which satisfy the equation $ABCB^{-1} - B = A$. Express C in terms of A and B. Show all your work.

$ABCB^{-1} - B = A$ / Add B to both sides

$ABCB^{-1} = A + B$ / mult by B on right

$ABC = AB + B^2$ / mult by A^{-1} on left.

$BC = B + A^{-1}B^2$ / mult by B^{-1} on left

$\boxed{C = I + B^{-1}A^{-1}BB}$ Also $B^{-1}A^{-1}(A+B)B$

If they complete $A^{-1} = \begin{pmatrix} 1/2 & 0 \\ -3/4 & 1/2 \end{pmatrix}$ 3/12

3. (18 points) Recall that two $n \times n$ matrices A and B are said to commute, if $AB = BA$.

12 points

(a) Find all 2×2 matrices, which commute with the matrix $A = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$.

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$AB = BA$$

$$\begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2a & 2b \\ (3a+2c) & (3b+2d) \end{pmatrix} = \begin{pmatrix} (2a+3b) & 2b \\ (2c+3d) & 2d \end{pmatrix} \Leftrightarrow \begin{cases} \cancel{2a} = \cancel{2a} + 3b \\ 2b = 2b & \text{redundent} \\ 3a+2c = 2c+3d \\ 3b + \cancel{2d} = \cancel{2d} & \text{redundent} \end{cases}$$

$$2a = 2a + 3b \Rightarrow \boxed{b = 0}$$

$$3a + 2c = 2c + 3d \Rightarrow (a - d = 0$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\boxed{\cancel{c, d \text{ are free variables}}}$

$$B = \left\{ \begin{pmatrix} d & 0 \\ c & d \end{pmatrix} : c, d \text{ are real numbers} \right\}$$

6 points

(b) Let A and B be two $n \times n$ matrices. Show that if A commutes with B and B is invertible, then A commutes with B^{-1}.

$AB = BA$ Multiply both sides by B^{-1} on the rig

$A = BAB^{-1}$ Multiply both sides by B^{-1} on the left

$B^{-1}A = AB^{-1}$

Hence, A and B^{-1} commute.

4. (17 points) Let A be an $m \times n$ matrix, $\vec{b}$ a non-zero vector in $\mathbb{R}^n$, $\vec{x}_1$ a solution of the equation $A\vec{x} = \vec{b}$, and $\vec{x}_h$ a solution of the equation $A\vec{x} = \vec{0}$.

(a) Show that $\vec{x}_1 + \vec{x}_h$ is a solution of the equation $A\vec{x} = \vec{b}$.

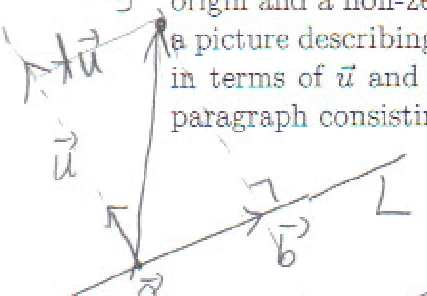
$$A(\vec{x}_1 + \vec{x}_h) = \underbrace{A\vec{x}_1}_{\vec{b}} + \underbrace{A\vec{x}_h}_{\vec{0}} = \vec{b} + \vec{0} = \vec{b}$$

(b) Let $\vec{x}_2$ be another solution of the system $A\vec{x} = \vec{b}$. Show that $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.

$$A(\vec{x}_2 - \vec{x}_1) = A\vec{x}_2 - A\vec{x}_1 = \vec{b} - \vec{b} = \vec{0}.$$

$\underbrace{\vec{b}}_{\vec{x}_1} + \underbrace{t\vec{u}}_{\vec{x}_h}$

(c) Let A be the 2×2 matrix of the projection of $\mathbb{R}^2$ onto a line L through the origin and a non-zero vector $\vec{b}$. Let $\vec{u}$ be a unit vector orthogonal to L. Draw a picture describing geometrically the set of solutions $\vec{x}$ of the system $A\vec{x} = \vec{b}$, in terms of $\vec{u}$ and $\vec{b}$. Then use your work above to justify the picture in a paragraph consisting of complete sentences. (⊥) Parts 4a and 4b



The set of soln of $A\vec{x} = \vec{0}$ is this line

$A\vec{u} = \vec{0}$

$A\vec{b} = \vec{b}$

← set of solutions of $A\vec{x} = \vec{b}$ is this line. $A\vec{b} = \vec{b}$, so $\vec{x}_1 = \vec{b}$ is a soln of $A\vec{x} = \vec{b}$. Any soln $\vec{x}_h$ of $A\vec{x} = \vec{0}$ is of the form $\vec{x}_h = t\vec{u}$. Parts (a) and (b) show that every soln $\vec{x}$ of $A\vec{x} = \vec{b}$ has the form $\vec{x} = \vec{x}_1 + \vec{x}_h = \vec{b} + t\vec{u}$.

The set of vectors of the form $\vec{b} + t\vec{u}$ is precisely the line in the diagram.

5. (20 points) Let L be the line in $\mathbb{R}^2$ through the origin and the vector $\vec{v} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

Recall that the reflection $Ref_L : \mathbb{R}^2 \to \mathbb{R}^2$ is given by the formula

7 pto

$$Ref_L(\vec{x}) \quad = \quad \frac{2(\vec{x}\cdot\vec{v})}{\vec{v}\cdot\vec{v}}\vec{v} - \vec{x}. \tag{1}$$

(a) Use the formula (1) to find the standard matrix A of Ref_L. $= (\vec{a_1}\ \vec{a_2})$

$$a_1 = A\begin{pmatrix}1\\0\end{pmatrix} = \frac{2\cdot\overbrace{\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}}}{\underbrace{\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}}_{4}}\cdot\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \sqrt{3}/2\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1/2\\ \sqrt{3}/2\end{pmatrix}$$

$$\vec{a}_2 = A\begin{pmatrix}0\\1\end{pmatrix} = \frac{2\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}}{4}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \frac{\cancel{2}\sqrt{3}}{\cancel{4}_{2}}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix}$$

$$A = \begin{pmatrix}-1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix}$$

3 pto

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plane about the origin $\frac{\pi}{3}$ radians (i.e., 60 degrees) counter-clockwise. Find the standard matrix B of the rotation T. Hint: $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$.

$$B = \begin{pmatrix}\cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3})\\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3})\end{pmatrix} = \begin{pmatrix}1/2 & -\sqrt{3}/2\\ +\sqrt{3}/2 & 1/2\end{pmatrix}$$

3 pto

(c) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $S(\vec{x}) = Ref_L(T(\vec{x}))$ (i.e., rotation followed by reflection). Express the standard matrix C of S in terms of the matrices A of Ref_L and B of T.

$C =$ AB.

3 pts

(d) Use the expression in part 5c to compute the matrix C. Note: The answer is $C = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

$$C = AB = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} =$$

$$= \begin{pmatrix} [-1/4 + 3/4] & [\sqrt{3}/4 + \sqrt{3}/4] \\ \sqrt{3}/4 + \sqrt{3}/4 & [-3/4 + \frac{1}{4}] \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -\frac{1}{2} \end{pmatrix}$$

4 pts

(e) Let $\tilde{L}$ be the line through the origin and the vector $\vec{w} = \cancel{\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}}$ $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$. The matrix C in part 5d is the matrix of the reflection $Ref_{\tilde{L}}$ with respect to this new line $\tilde{L}$. You need **not** prove this fact. Use this fact and your work above in order to express the rotation T in terms of the reflections Ref_L and $Ref_{\tilde{L}}$. $(\vec{x})$

Method I:

$T(\vec{x}) =$ $Ref_L^{-1}(Ref_{\tilde{L}}(\vec{x}))$ $= Ref_L(Ref_{\tilde{L}}(\vec{x}))$. Justify your answer! $\Leftarrow S(\vec{x}) = Ref_L(T(\vec{x}))$

$\boxed{\text{Since } Ref_L^{-1} = Ref_L}$

Method II: The matrix of T is B.

$$B = A^{-1}C$$

So $T(\vec{x}) = B\vec{x} = A^{-1}C\vec{x} = Ref_L^{-1}(Ref_{\tilde{L}}(\vec{x})) = Ref_L(Ref_{\tilde{L}}(\vec{x}))$

6. (15 points)

5 points

(a) Is the vector $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ a linear combination of the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$?

Justify your answer!

$$\left(\begin{array}{cc|c} 1 & 4 & 2 \\ 2 & 3 & 3 \\ 3 & 2 & 4 \end{array}\right) \xrightarrow[\text{Add } -3R_1 \text{ to } R_3]{\text{Add } -2R_1 \text{ to } R_2} \left(\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & -5 & -1 \\ 0 & -10 & -2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{array}\right)$$

Yes, we do not have a pivot in the rightmost column, so the eq $\begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ is consistent.

5 points

(b) Let A be a 4×3 matrix such that the system $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ has a unique solution.

i. What is the rank of A? Justify your answer!

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \cdot & \cdot \\ a_{22} & \cdot & \cdot \\ a_{41} & \cdot & a_{43} \end{pmatrix}$$

$\text{rank}(A) = 3$.

Reason: $\text{Rank}(A) \leq 3$. If $\text{rank}(A) < 3$, and $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ is consistent, then it will have ∞ many solutions (due to a free variable).

5 points

ii. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$. Is the image of T equals the whole of $\mathbb{R}^4$? Justify your answer!

No, since A does not have a pivot in every row.