TEST 1: Answers

Math 235
Fall 2006

Name: $\underbrace{\hspace{6cm}}$

Read all of the following information before starting the exam:

- Do all work to be graded in the space provided. If you need extra space, use the reverse of the page and indicate on the front that you have continued the work on the back.
- *You must support your answers with necessary work. My favorite number is three. Unsupported answers will receive zero credit.*
- Circle or otherwise indicate your final answers.
- You do not need to simplify expressions such as $\sqrt{11}$, e^2, π with decimal approximations.
- This test has 9 problems and is worth 100 points, plus some extra credit at the end. It is your responsibility to make sure that you have all of the pages!
- If you have read of all these instructions, check this box for one bonus point $\square$.
- Good luck!

1. (*2 points*) Go back and read the instructions on page one. What is my favorite number?

2. (*12 points*) Solve the system of linear equations:

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 0 & 2 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Use Gaussian Elimination to get the solution

$$\vec{x} = \begin{pmatrix} 7 \\ \frac{5}{2} \\ -3 \end{pmatrix}.$$

3. (*10 points*) Let C be a 5×7 matrix (5 rows and 7 columns). Place a check in the correct box identifying each object.

Object	Subspace of $\mathbb{R}^5$	Subspace of $\mathbb{R}^7$	Not a subspace
Solutions to $C\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$		$\surd$	
Solutions to $C\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 4 \\ 3 \end{pmatrix}$			$\surd$
$\ker(C)$		$\surd$	
$\mathrm{im}(C)$	$\surd$		
My dog Duey			$\surd$

Note that C gives a linear transformation from $\mathbb{R}^7 \to \mathbb{R}^5$. Thus the $\mathrm{im}(C)$ is a subspace of $\mathbb{R}^5$ and $\ker(C)$ is a subspace of $\mathbb{R}^7$. One can check that the solution set to $C\vec{x} = \vec{b}$ for $\vec{b} \neq \vec{0}$ does not contain $\vec{0}$, and hence is not a subspace. My dog Duey does not have a vector addition defined on him, and hence cannot be a subspace.

4. (*14 points*) Suppose A is a square $n \times n$ matrix, and A similar to B. Write True or False for the following statements.

a. (*2 pts*) True B is a $n \times n$ matrix.

b. (*2 pts*) True There exists an invertible $n \times n$ matrix S such that $B = S^{-1}AS$.

c. (*2 pts*) True B is similar to A.

d. (*2 pts*) True If $A^3 + 2A + I$ is the zero matrix, then $B^3 + 2B + I$ is the zero matrix.

e. (*2 pts*) True A^4 is similar to B^4.

f. (*2 pts*) True If $\text{rank}(A) = n$, then B is invertible.

g. (*2 pts*) True If $\dim(\ker(A)) = 5$, then $\dim(\ker(B)) = 5$.

5. (*14 points*) Consider the subspace $V \subseteq \mathbb{R}^3$ of vectors perpendicular to $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

a. (*4 pts*) Find a matrix A such that $V = \ker(A)$.

Note that $\vec{x} \in V$ if and only if $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \vec{x} = 0$. In particular, V is the set of solutions to $x_1 - x_3 = 0$.

It follows that $V = \ker(A)$, where $A = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$.

b. (*10 pts*) Let $\mathcal{B}$ be the ordered basis

$$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right),$$

and let T be the linear transformation of projection to V. Compute $\text{Mat}_{\mathcal{B}}^{\mathcal{B}}(T)$.

Let $\vec{f}_1, \vec{f}_2, \vec{f}_3$ denote the vectors of $\mathcal{B}$. Then

$$\text{Mat}_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{bmatrix} [T\vec{f}_1]_{\mathcal{B}} & [T\vec{f}_2]_{\mathcal{B}} & [T\vec{f}_3]_{\mathcal{B}} \end{bmatrix}.$$

Since $\vec{f}_1 \in V$, $T\vec{f}_1 = \vec{f}_1$. Expressing in $\mathcal{B}$-coordinates,

$$[T\vec{f}_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\vec{f}_2 \perp V$, $T\vec{f}_2 = \vec{0}$. Expressing in $\mathcal{B}$-coordinates,

$$[T\vec{f}_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\vec{f}_3 \in V$, $T\vec{f}_3 = \vec{f}_3$. Expressing in $\mathcal{B}$-coordinates,

$$[T\vec{f}_3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It follows that

$$\text{Mat}_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. (*12 points*) Let $T := \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ 2x_2 + x_3 \\ x_1 + x_3 \end{pmatrix}.$$

a. (*6 pts*) Express T as a matrix A.

By inspection, $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

b. (*6 pts*) Compute the dimension of the image of T.

Recall that $\dim(\text{im}(T)) = \text{rank}(A)$, and so it suffices to compute the rank of A. Use Gaussian Elimination to show that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ and hence $\dim(\text{im}(T)) = 2$.

7. (*10 points*) Let $T : \mathbb{R}^{12} \to \mathbb{R}$ be a linear transformation.

a. (*5 pts*) What are the possible dimensions of $\text{im}(T)$? Justify.

Since $\text{im}(T)$ is a subspace of $\mathbb{R}$, it follows that $\dim(\text{im}(T)) = 0$ or 1.

b. (*5 pts*) What are the possible dimensions of $\ker(T)$? Justify.

The Rank-Nullity Theorem gives that $\dim(\text{im}(T)) + \dim(\ker(T)) = 12$. By above, $\dim(\text{im}(T)) = 0$ or 1, so it follows that $\dim(\ker(T)) = 11$ or 12.

8. (*14 points*) Let $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$. Then $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a. (*7 pts*) Find a *basis* for the image of A.

The linear relations between the columns of $\text{rref}(A)$ are precisely the linear relations between the columns of A. It follows that the first two columns of A span the image of A and are linearly independent, and so a basis of the image of A is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ -1 \end{pmatrix} \right\}$.

b. (*7 pts*) Find a *basis* for the kernel of A.

Note that $\ker(A) = \ker(\text{rref}(A))$. The first two columns of $\text{rref}(A)$ have pivots, so x_1 and x_2 can be solved for explicitly. The third and fourth columns do not contain pivots, hence x_3 and x_4 are free. It follows that if $x_3 = s$ and $x_4 = t$, then $x_1 = -2s - 4t$ and $x_2 = 3s + t$. Thus a basis for $\ker(A)$ is given by $\left\{ \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

9. (*12 points*) Let $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Then $\mathcal{B} = (\vec{u}, \vec{v})$ is an ordered basis of the subspace $V = \operatorname{span} \mathcal{B} \subseteq \mathbb{R}^4$. Find the vector $\vec{w} \in V$ such that $[\vec{w}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$\vec{w} = 1\vec{u} + 2\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}.$$

Scratch Paper

BONUS Survey: 1 bonus point total for completion

1. Suppose you are running in a race. You pass the second place runner. What place are you in now?

   Second place.

2. Out of 100 points, what do you think is your score on this test?

   95

3. Out of 100 points, what do you think is the class average on this test?

   70