Math 235 Midterm 2 Spring 2008

Name:____________________________

Justify all your answers. Show all your work!!!

1. (20 points) You are given below the matrix A together with its row reduced echelon form B

$$A = \begin{pmatrix} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 4 & 2 & 2 & 2 \\ 2 & 1 & 4 & -1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note: you do **not** have to check that A and B are indeed row equivalent.

a) Determine the rank of A, $\dim(\ker(A))$, and $\dim(\text{im}(A))$. Explain how these are determined by the matrix B.

b) Find a basis for the kernel $\ker(A)$ of A.

c) Find a basis for the image $\text{im}(A)$ of A.

d) Does the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ belong to the image of A? Use part c to minimize your computations. **Justify** your answer!

2. (12 points) Let A be a 4×5 matrix with columns $\vec{a}_1, \ldots, \vec{a}_5$. We are given that the vector $\begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 5 \end{pmatrix}$ belongs to the kernel of A and the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ span the image of A.

a) Express $\vec{a}_5$ as a linear combination of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4$.

b) Determine $\dim(\mathrm{im}(A))$. Justify your answer.

c) Determine $\dim(\ker(A))$. Justify your answer.

3. (20 points) Let $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\beta := \{v_1, v_2\}$ the corresponding basis of $\mathbb{R}^2$.

   (a) Find a vector w in $\mathbb{R}^2$, such that the coordinate vector of w with respect to the basis β is $[w]_\beta = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

   (b) Let $w_1 := \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $w_2 := \begin{pmatrix} -3 \\ -4 \end{pmatrix}$. Find the coordinate vectors $[w_1]_\beta$ and $[w_2]_\beta$ with respect to the basis β.

   (c) Let $A = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ the linear transformation given by $T(\vec{x}) = A\vec{x}$. Note that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Use this information and your work in part 3b to find the matrix B of T with respect to the basis β of $\mathbb{R}^2$.

(d) Let $\tilde{v}_1$, $\tilde{v}_2$, be two linearly independent vectors in $\mathbb{R}^2$, and $\widetilde{S} := (\tilde{v}_1 \tilde{v}_2)$ the 2×2 matrix with $\tilde{v}_j$ as its j-th column. Let $\widetilde{B}$ be the matrix of the linear transformation T in part 3c, with respect to the new basis $\tilde{\beta} := \{\tilde{v}_1, \tilde{v}_2\}$. Express $\widetilde{B}$ in terms of the matrices A and $\widetilde{S}$.

(e) Let $S := (v_1 v_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Express $\widetilde{B}$ in terms of the matrices S, $\widetilde{S}$, and B. Your final answer should not involve the matrix A. Hint: Express first A in terms of S and B. Then express A in terms of $\widetilde{S}$ and $\widetilde{B}$.

4. (18 points) Denote the vector space of 2×2 matrices by $R^{2\times 2}$. Let $A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ the linear transformation given by $T(M) = AM - MA$.

(a) Find the matrix B of T in the basis
$\beta := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $R^{2\times 2}$.

(b) Find a basis for $\ker(B)$.

(c) Find a basis for $\ker(T)$.

(d) Find a basis for $\mathrm{im}(B)$.

(e) Find a basis for $\mathrm{im}(T)$.

5. (10 points) Let V and W be two vector spaces and $T : V \to W$ a linear transformation from V to W. Let p be a positive integer and $\{f_1, \ldots, f_p\}$ a linearly **dependent** subset of V consisting of p elements. Show the the subset $\{T(f_1), \ldots, T(f_p)\}$ of W is linearly dependent as well. Note: Provide an argument that works for general vector spaces, starting with the definition of linear dependence.

6. (20 points) Let $C^\infty(\mathbb{R})$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, having derivatives of all orders. Denote by V the subspace of $C^\infty(\mathbb{R})$ spanned by the functions $f_1(x) = e^x$, $f_2(x) = e^{2x}$, and $f_3(x) = e^{3x}$. Let $T : V \to \mathbb{R}^3$ be the transformation given by $T(f) := \begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix}$.

   (a) Show that the transformation T is linear. In other words, verify the following identities, for any two elements f, g of V, and for every scalar k.

      i. $T(f+g) = T(f) + T(g)$.

      ii. $T(kf) = kT(f)$.

   (b) Show that the subset $\{T(f_1), T(f_2), T(f_3)\}$ of $\mathbb{R}^3$ is linearly independent. Hint: Recall that the chain rule yields $(e^{2x})' = 2e^{2x}$, $(e^{2x})'' = 2^2e^{2x}$, and so $f_2''(0) = 4$.

(c) Show that $\mathrm{im}(T)$ is the whole of $\mathbb{R}^3$.

(d) Show the the subset $\{e^x, e^{2x}, e^{3x}\}$ of V is linearly independent. Hint: Use part 6b and question 5.

(e) Show that $T : V \to \mathbb{R}^3$ is an isomorphism.