Math 235 section 1 Midterm 1 Spring 2008

Name:__________________________

1. (15 points) a) Show that the row **reduced** echelon form of the augmented matrix of the system
$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 4$
$x_2 - x_3 + x_4 + x_5 = 4$
$2x_1 + 4x_3 + 3x_4 + 5x_5 = 2$
is $\begin{pmatrix} 1 & 0 & 2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$. Use at most five elementary operations. Show all your work. Clearly write in words each elementary row operation you used.

b) Find the general solution for the system.

2. (20 points) You are given that the row reduced echelon form of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 & 0 & 4 \\ 1 & 0 & 2 & -1 & 2 & 0 \end{pmatrix} \text{ is } B = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

You do **not** need to verify this statement.

(a) Write the general solutions of the system $A\vec{x} = \vec{0}$ in parametric form $\vec{x} = (\text{first free variable})\vec{v}_1 + (\text{second free variable})\vec{v}_2 + \dots$

(b) Let $T : \mathbb{R}^6 \to \mathbb{R}^4$ be the linear transformations given by $T(\vec{x}) = A\vec{x}$. Find a basis for the kernel $\ker(T)$. In other words, find a linearly independent set of vectors in $\ker(T)$, which spans $\ker(T)$. Explain why the set you found is linearly independent, and why it spans $\ker(T)$.

(c) For each vector in the basis you found in part 2b, write down a corresponding linear relation among the columns of the original matrix A (use the notation a_i for the i-th column). Then use each of these relations to find a redundent vector among the columns of A (i.e., a column vector $\vec{a}_i$, which is a linear combination of the preceeding columns $\vec{a}_1, \ldots, \vec{a}_{i-1}$).

(d) Is the image of T equal to the whole of $\mathbb{R}^4$? Justify your answer.

3. (a) (7 points) Let A, B, C be $n \times n$ matrices, with A invertible, which satisfy the equation $A(C + I_n)A^{-1} = B$, where I_n is the $n \times n$ identity matrix. Express C in terms of A and B. Show all your work.

(b) (8 points) Let A be an $n \times n$ matrix satisfying $A^3 + 5A^2 + 2A - I_n = 0$, where 0 is the $n \times n$ matrix all of which entries are zero. Show that A is invertible and express A^{-1} in terms of A.
Hint: Rewrite the equation as $A^3 + 5A^2 + 2AI_n - I_n = 0$.

4. (15 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation with standard matrix A. Assume that there exists a unique vector $\vec{x}$ in $\mathbb{R}^3$, such that $T(\vec{x}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

Carfully **justify** your answers to the following questions.

(a) The rank of A is: ____.

(b) Describe geometrically the kernel of T.

(c) Is it true that the equation $A\vec{x} = \vec{y}$ has a unique solution $\vec{x}$, for every vector $\vec{y}$ in $\mathbb{R}^4$? Justify!

5. (a) (8 points) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$.

(b) (7 points) Find the set of all matrices B, satisfying the matrix equation $BA = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

6. (20 points) Let L_θ be the line in $\mathbb{R}^2$ through the origin and the unit vector $\vec{u} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$. Recal that the reflection $Ref_{L_\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane about the line L_θ is given by the formula $Ref_{L_\theta}(\vec{x}) = 2\,(\vec{u} \cdot \vec{x})\,\vec{u} - \vec{x}$.

(a) Use the algebraic properties of the dot product to show that Ref_{L_θ} is a linear transformation. In other words, verify the following identities, for any two vectors $\vec{v}, \vec{w}$ and for every scalar k.

i. $Ref_{L_\theta}(\vec{v} + \vec{w}) = Ref_{L_\theta}(\vec{v}) + Ref_{L_\theta}(\vec{w})$.

ii. $Ref_{L_\theta}(k\vec{v}) = kRef_{L_\theta}(\vec{v})$.

(b) Use the above formula for $Ref_{L_\theta}(\vec{x})$ to show that the standard matrix A of Ref_{L_θ} is $A = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$. Hint: Recall the identities:
$\cos^2(\theta) + \sin^2(\theta) = 1$, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$, $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the composition $T(\vec{x}) = Ref_{L_\phi}(Ref_{L_\theta}(\vec{x}))$, where L_ϕ is the line through the origin and $\vec{w} = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix}$. Express the standard matrix C of T in terms of the standard matrices A of Ref_{L_θ} and B of Ref_{L_ϕ}: $C =$ _____.

Use this expression to show the equality $C = \begin{pmatrix} \cos(2\phi - 2\theta) & -\sin(2\phi - 2\theta) \\ \sin(2\phi - 2\theta) & \cos(2\phi - 2\theta) \end{pmatrix}$.

Hint: Recall the identities $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ and $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$.

(d) The linear transformation T in part 6c is described more directly as the rotation of the plane about the origin an angle _____ counterclockwise. **Justify** your answer.