Math 235 Section 1 Final Exam Spring 2008

Justify all your answers. Show all your work!!!

1. (10 points) The matrices A and B below are row equivalent (you do **not** need to check this fact).

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 & -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find a basis for $\ker(A)$.

b) Find a basis for image(A).

2. (16 points) Consider the matrix $A = \begin{pmatrix} -1 & -2 & -4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix}$.

(a) Show that the characteristic polynomial of A is $-(\lambda-1)(\lambda+1)(\lambda-2)$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

(c) Find an invertible matrix S and a diagonal matrix D such that the matrix A above satisfies $S^{-1}AS = D$

3. (16 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .7 & .3 \\ .3 & .7 \end{pmatrix}$.

(a) The eigenvalue of v_1 is ______
The eigenvalue of v_2 is ______

(b) Set $w := \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the coordinate vector $[w]_\beta$ of w in the basis $\beta := \{v_1, v_2\}$.

(c) Compute $A^{100}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(d) As n gets larger, the vector $A^n\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ approaches _____. Justify your answer.

4. (16 points) Let V be the plane in $\mathbb{R}^3$ spanned by $v_1 := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $v_2 := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

(a) Find the orthogonal projection $\text{proj}_V(w)$ of $w = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$ into V.

(b) Write w as a sum of a vector in V and a vector orthogonal to V.

(c) Find the distance from w to V (i.e., to the vector in V closest to w).

5. (16 points)

(a) Let A and S be two $n \times n$ matrices with real coefficients with S invertible. Then the columns $v_1, \ldots, v_n$ of S form a basis of $\mathbb{R}^n$. Complete the following sentence: The matrix $S^{-1}AS$ is diagonal with d_i as its (i,i)-entry, if and only if for all $1 \leq i \leq n$, the vector v_i is

____.

(b) For what values of θ is the matrix $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ diagonalizable? I.e., for what values of θ does there exist some invertible 2×2 matrix S with real coefficients, such that $S^{-1}AS$ is diagonal? Justify your answer!

(c) For what values of k is the matrix $\begin{pmatrix} 2 & 0 \\ k & 2 \end{pmatrix}$ diagonalizable? Justify your answer!

6. (10 points) Let V be the subspace of $\mathbb{R}^4$ spanned by

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 3 \\ 1 \\ 3 \\ -1 \end{pmatrix}.$$

(a) Use the Gram-Schmidt process to find an orthonormal basis for V.

(b) Find a basis for the orthogonal complement $V^{\perp}$ of V in R^4.

7. (16 points) Let P_3 be the vector space of polynomials of degree ≤ 3 with real coefficients. Let $T : P_3 \to \mathbb{R}^4$ be the linear transformation given by

$$T(f) = \begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \end{pmatrix}.$$ Consider the following four polynomials in P_3:

$f_1(x) = \frac{-1}{6}(x-2)(x-3)(x-4)$, $\quad f_2(x) = \frac{1}{2}(x-1)(x-3)(x-4)$,
$f_3(x) = \frac{-1}{2}(x-1)(x-2)(x-4)$, $\quad f_4(x) = \frac{1}{6}(x-1)(x-2)(x-3)$.

Let $U : \mathbb{R}^4 \to P_3$ be the linear transformation given by

$$U\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x).$$

(a) Show that the composition $TU : \mathbb{R}^4 \to \mathbb{R}^4$ is the identity linear transformation. In other words, show that $T(U(\vec{x})) = \vec{x}$, for all $\vec{x}$ in $\mathbb{R}^4$.

(b) Show that T is an isomorphism. Hint: Show first that image$(T) = \mathbb{R}^4$.

(c) Show that the set $\{f_1, f_2, f_3, f_4\}$ is a basis of P_3. Use the previous parts to minimize your calculations.

(d) Find a polynomial $g(x)$ of degree ≤ 3 satisfying $g(1) = 2$, $g(2) = 3$, $g(3) = 5$, $g(4) = 7$. Hint: Express g as a linear combination of the f_i's. You need not simplify your answer.