Math 235H Written homework assignment 6 Fall 2020

- **Justify all your answers!**
- **Please follow carefully the instructions on the document titled "Submitting PDF documents via Gradescope" (see the PDF file attachment to the announcement "Gradescope - our hand written homework submission platform" on our Moodle page). After you upload the PDF file to Gradescope you will need also to respond to the Gradescope prompt asking you to indicate, for each of the problems, the page in which the solution to that problem is found. If you have solutions to more than one problem in a page, make sure that the problem number is large and clear.**

Definition: An $n \times n$ matrix is called *nilpotent* if $A^m = 0$ for some positive integer m, where the zero on the right hand side is the $n \times n$ matrix all of which entries are zero.

This homework assignment deals with *nilpotent* matrices. Such matrices are important due to the following theorem of Jordan (dealing with non-diagonalizable matrices).

Jordan's Decomposition Theorem: *Let A be a square $n \times n$ matrix. There exists a unique pair of $n \times n$ matrices D and N, such that 1) $A = D + N$, 2) the matrix D is diagonalizable,*[1] *3) the matrix N is nilpotent, and 4) D and N commute, $DN = ND$. If A has real entries, so do D and N.*

Example: Let λ_1 and λ_2 be two distinct scalars and let $A = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{pmatrix}$.

Then $D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. If B is an $n \times n$ matrix similar to A, i.e., $B = P^{-1}AP$ for some invertible matrix P, then the Jordan decomposition of B is $B = P^{-1}AP = P^{-1}DP + P^{-1}NP$, where $P^{-1}DP$ is diagonalizable and $P^{-1}NP$ is nilpotent.

[1] There exists an invertible matrix P such that $P^{-1}DP$ is diagonal. If A has real entries and the characteristic polynomial of A factors as a product of linear terms with real roots, then P may be chosen with real entries. Otherwise, the entries of P are complex numbers.

The above is mentioned only for motivation. You will not need to use the Jordan decomposition theorem below.

1. (a) Show that 0 is the only eigenvalue of a nilpotent $n \times n$ matrix A. Hint: Assume that λ is the eigenvalue of some (necessarily non-zero) eigenvector $\vec{v}$, compute $A^m\vec{v}$ in two ways, and use it to show that $\lambda = 0$.

   (b) Use part 1a to show that if A is a nilpotent $n \times n$ matrix, then its characteristic polynomial is $f(x) = (-1)^n x^n$. Remark: The converse is also true, if the characteristic polynomial is $f(x) = x^n$, then A is nilpotent. This follows from the Cayley-Hamilton Theorem (taught in Math 545):
   Theorem: *Let A be an $n \times n$ matrix with characteristic polynomial $f(x) = (-1)^n x^n + c_{n-1}x^{n-1} + \cdots + c_0$. Then*

   $$f(A) = (-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_0 I$$

   is the zero matrix.

   Do **not** use the Cayley-Hamilton Theorem in your solutions to any of the problems in this homework assignment.

   (c) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Recall that the trace $\mathrm{tr}(A)$ is the sum $a + d$ of the diagonal entries of A. Show that A is nilpotent if and only if $\mathrm{tr}(A) = 0$ and $\det(A) = 0$. Hint: Compute the characteristic polynomial of A.

   (d) Set $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Show that A is nilpotent. Then show that $P^{-1}AP$ is nilpotent for every 3×3 invertible matrix P.

2. An $n \times n$ matrix A is said to be *strictly upper triangular*, if all entries on or below the diagonal vanish, i.e., $a_{i,j} = 0$, if $i \geq j$. Show, without using the Cayley-Hamilton theorem, that every strictly upper triangular $n \times n$ matrix A is nilpotent. Hint: Let H_k be the subspace span$\{e_1, \ldots e_k\}$ of $\mathbb{R}^n$, $1 \leq k \leq n$, and set $H_0 = 0$. We get the increasing sequence of subspaces

   $$0 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_{n-1} \subset H_n = \mathbb{R}^n.$$

   Show that A takes every vector in H_k into H_{k-1}, for $k \geq 1$. In particular, $Ae_1 = 0$. Use it to show that the i-th power A^i takes every vector in H_k into H_{k-i}, for $k \geq i + 1$, and H_i is contained in $Null(A^i)$. Conclude that $A^n = 0$.

3. Consider an $n \times n$ nilpotent matrix A and let m be the smallest positive integer, such that $A^m = 0$. Choose a vector $\vec{v}$ in $\mathbb{R}^n$, such that $A^{m-1}\vec{v} \neq \vec{0}$. Show that the vectors $\vec{v}$, $A\vec{v}$, ..., $A^{m-1}\vec{v}$ are linearly independent.
   Hint: Assume that $c_0\vec{v} + c_1 A\vec{v} + \ldots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$. Multiply both sides by A^{m-1} to show that $c_0 = 0$. Next show that $c_1 = 0$ and so on.

4. Let A be a nilpotent $n \times n$ matrix. Use problem 3 to show that $A^n = 0$ (without using the Cayley-Hamilton theorem).

5. Let A be a 4×4 nilpotent matrix such that $A^3\vec{v} \neq \vec{0}$, for some vector $\vec{v}$ in $\mathbb{R}^4$.

   (a) Show that $\mathcal{B} := \{\vec{v}, A\vec{v}, A^2\vec{v}, A^3\vec{v}\}$ is a basis of $\mathbb{R}^4$ and compute the $\mathcal{B}$-matrix (as in HW5 Problem 6(b)) of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ with standard matrix A.

   (b) Let $\mathcal{C}$ be the basis $\{A^3\vec{v}, A^2\vec{v}, A\vec{v}, \vec{v}\}$ of $\mathbb{R}^4$ (same set of vectors as $\mathcal{B}$, but in reversed order). Compute the $\mathcal{C}$-matrix of T.