1. An $n \times n$ matrix is called *nilpotent* if $A^m = 0$ for some positive integer m, where the zero on the right hand side is the $n \times n$ matrix all of which entries are zero.

   (a) Show that 0 is the only eigenvalue of a nilpotent $n \times n$ matrix A. Hint: Assume that λ is the eigenvalue of some (necessarily non-zero) eigenvector $\vec{v}$, compute $A^m\vec{v}$ in two ways, and use it to show that $\lambda = 0$.

Assume $A^m = 0$. Then $A^m \vec{v} = \vec{0}$. On the other hand, if λ is the eigenvalue of $\vec{v}$, then $A^m(\vec{v}) = A(A \cdots (\underbrace{A(v)}_{\lambda v}) \cdots) = \lambda^m \vec{v}$. Thus, $\vec{0} = \lambda^m \vec{v}$. The equality $\lambda = 0$ follows, since $\vec{v} \neq \vec{0}$.

   (b) Use part 1a to show that if A is a nilpotent $n \times n$ matrix, then its characteristic polynomial is $f(x) = x^n$.

The characteristic poly of an $n \times n$ matrix has degree n, leading coeff x^n and its roots are eigenvalues, so the only root of $f(x)$ is $x = 0$. The only such poly is $f(x) = x^n$, since if $f(x)$ is not a scalar multiple of a power of x, then f has a non-zero (complex possibly) root.

   Remark: The converse is also true, if the characteristic polynomial is $f(x) = x^n$, then A is nilpotent. This follows from the Cayley-Hamilton **Theorem:** *Let A be an $n \times n$ matrix and $f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$ its characteristic polynomial. Then $f(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_0 I$ is the zero matrix.*

   (c) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Recall that the trace $\mathrm{tr}(A)$ is the sum $a + d$ of the diagonal entries of A. Show that A is nilpotent if and only if $\mathrm{tr}(A) = 0$ and $\det(A) = 0$. Hint: Compute the characteristic polynomial of A.

$\det(A - xI) = \det\begin{pmatrix} a-x & b \\ c & d-x \end{pmatrix} = x^2 - (a+d)x + (ad - bc) =$
$= x^2 - \mathrm{tr}(A)x + \det(A)$. On the other hand, $\det(A - xI) = x^2$, since A is nilpotent.
Hence, $\mathrm{tr}(A) = 0$ and $\det(A) = 0$. Conversely, if $\mathrm{tr}(A) = 0$, then $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ and $A^2 = \begin{pmatrix} a^2+bc & 0 \\ 0 & cb+a^2 \end{pmatrix} = \begin{pmatrix} -\det(A) & 0 \\ 0 & -\det(A) \end{pmatrix}$, so $A^2 = 0$ if $\det(A) = 0$ as well.

(d) Set $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Show A is nilpotent. Then show that $P^{-1}AP$ is nilpotent for every 3×3 invertible matrix P.

$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A^3 = 0$.

Thus, $(P^{-1}AP)^3 = (P^{-1}A\underbrace{P)(P^{-1}}_{\text{cancel}}A\underbrace{P)(P^{-1}}_{\text{cancel}}AP) =$

$= P^{-1}A^3P = P^{-1}0P = 0.$

2 An $n \times n$ matrix is called *nilpotent* if $A^m = 0$ for some positive integer m, where the zero on the right hand side is the $n \times n$ matrix all of which entries are zero. An $n \times n$ matrix A is said to be *strictly upper triangular*, if all entries on or below the diagonal vanish, i.e., $a_{i,j} = 0$, if $i \geq j$.

(a) Show that every strictly upper triangular matrix A is nilpotent. Hint: Show that A takes span$\{e_1, \ldots e_k\}$ into span$\{e_1, \ldots e_{k-1}\}$, if $k \geq 2$, and $Ae_1 = 0$ and use it to show that $A^n = 0$.

$(\vec{a}_1 \vec{a}_2 \cdots \vec{a}_m)$

$\shortparallel$

If A is strictly upper triangular, then its j-th column $\vec{a}_j$ belongs to $\mathrm{Span}\{e_1, \ldots, e_{j-1}\}$, if $2 \leq j \leq m$, and $\vec{a}_1 = \vec{0}$. For $1 \leq j \leq m$, set $H_j := \mathrm{Span}\{e_1, \ldots, e_j\}$. Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be multiplication by A, so $T(\vec{x}) = A\vec{x}$. Then $T(H_j) = \{T(\vec{x})$ such that $\vec{x}$ is in $H_j\} =$
$= \{A\vec{x} : \vec{x} = \sum_{i=1}^{j} x_i e_i,\ x_i \text{ scalars}\} = \{\sum_{i=1}^{j} x_i \underbrace{Ae_i}_{\vec{a}_i} : x_i \text{ scalars}\} = \mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_j\}$
which is contained in H_{j-1}. So $T(H_j) \subset H_{j-1}$, for $1 \leq j \leq m$, where we define H_0 to be the zero subspace.
Given $\vec{x}$ in $\mathbb{R}^m = H_m$, we get that $A\vec{x} = T(\vec{x})$ is in H_{m-1}, $A^2\vec{x} = T(T(\vec{x}))$ is in $T(H_{m-1})$, so in H_{m-2}, and for $1 \leq d \leq m$ $A^d\vec{x} = T^d(\vec{x})$ is in H_{m-d}. So $A^m\vec{x}$ is in $H_0 = \{\vec{0}\}$, So $A^m\vec{x} = \vec{0}$ for all $\vec{x}$ in $\mathbb{R}^m$. So $A^m =$ zero matrix.

(b) Give an example of a 2×2 nilpotent matrix, which diagonal entries are non-zero.

$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

3 Consider an $n \times n$ nilpotent matrix A and let m be the smallest positive integer, such that $A^m = 0$. Choose a vector $\vec{v}$ in $\mathbb{R}^n$, such that $A^{m-1}\vec{v} \neq \vec{0}$. Show that the vectors $\vec{v}$, $A\vec{v}$, $\ldots$, $A^{m-1}\vec{v}$ are linearly independent.
Hint: Assume that $c_0\vec{v} + c_1 A\vec{v} + \ldots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$. (*) Multiply both sides by A^{m-1} to show that $c_0 = 0$. Next show that $c_1 = 0$ and so on.

Note first that $A^j \vec{v} \neq \vec{0}$ for $0 \le j \le m-1$, since $\vec{0} \neq A^{m-1}\vec{v} = A^{m-j-1}(A^j \vec{v})$. (We set $A^0 = I$).
Also, $A^k \vec{v} = 0$, for $k \geq m$, since $A^k\vec{v} = A^{k-m}(A^m v) = A^{k-m}\vec{0} = \vec{0}$.
Multiply both sides of (*) by A^{m-1} to get

$$c_0 A^{m-1}\vec{v} + c_1 \underbrace{A^m \vec{v}}_{\vec{0}} + \cdots + c_{m-1} \underbrace{A^{2(m-1)}}_{\substack{\| \\ \vec{0} \text{ if } m>1}} \vec{v} = \vec{0}.$$

Now $A^{m-1}\vec{v} \neq \vec{0}$.
Hence $\boxed{c_0 = 0}$. ~~Thus~~

If $m=1$, we are done. If $m>1$, then $A(c_1\vec{v} + c_2 A\vec{v} + \cdots + c_{m-1}A^{m-2}\vec{v}) = \vec{0}$

Multiply both sides by A^{m-2} we get $\vec{0} = c_1 A^{m-1}\vec{v} + \underbrace{A^m}_{0}(c_2\vec{v} + c_3 A\vec{v} + \cdots + c_{m-1}A^{m-3}\vec{v})$

Hence $\boxed{c_1 = 0}$. If $m=2$, we are done. otherwise we get
$A^2(c_2\vec{v} + \cdots + c_{m-1}A^{m-3}\vec{v}) = \vec{0}$.

Repeating this process, where in the j-th step we know that $A^{j-1}(c_{j-1}\vec{v} + c_j A\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v}) = \vec{0}$ and we multiply by A^{m-j} both sides to show that $\boxed{c_{j-1} = 0}$.
After $m-1$ steps we get that all $c_i = 0$.

Hence, the set $\{\vec{v}, A\vec{v}, \ldots, A^{m-1}\vec{v}\}$ is linearly independent.

4 Let A be a nilpotent $n \times n$ matrix. Show that $A^n = 0$. Hint: Use problem 3

Proof by contradiction.
Assume that $A^n \neq 0$. Then there exists a vector $\vec{v}$ in $\mathbb{R}^n$, such that $A^n\vec{v} \neq \vec{0}$ (e.g., if the j-th column of A is non-zero, take $\vec{v} = \vec{e}_j$, the j-th column of the $n \times n$ identity matrix). Then $\vec{v}, A\vec{v}, \ldots, A^n\vec{v}$ is a linearly independent set consisting of $n+1$ vectors, by Problem 6. But every set of $(n+1)$ vectors in $\mathbb{R}^n$ is linearly dependent. A contradiction. Hence $A^n = 0$.

15 Let A be a 4×4 nilpotent matrix such that $A^3\vec{v} \neq \vec{0}$, for some vector $\vec{v}$ in $\mathbb{R}^4$. Show that $\mathcal{B} := \{\vec{v}, A\vec{v}, A^2\vec{v}, A^3\vec{v}\}$ is a basis of $\mathbb{R}^4$ and compute the $\mathcal{B}$-matrix of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ with standard matrix A. B

$B = (\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4)$, where (using 8a)

$$\vec{b}_1 = [T(\vec{v})]_{\mathcal{B}} = [A\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{b}_2 = [T(A\vec{v})]_{\mathcal{B}} = [A^2\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{b}_3 = [T(A^2\vec{v})]_{\mathcal{B}} = [A^3\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now $A^4 = 0$, by Problem 7. Hence

$$\vec{b}_4 = [T(A^3\vec{v})]_{\mathcal{B}} = [A^4\vec{v}]_{\mathcal{B}} = [\vec{0}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

So, $$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$