Math 235H Written homework assignment 5 Fall 2020

- **Justify all your answers!**
- **Please follow carefully the instructions on the document titled "Submitting PDF documents via Gradescope" (see the PDF file attachment to the announcement "Gradescope - our hand written homework submission platform" on our Moodle page). After you upload the PDF file to Gradescope you will need also to respond to the Gradescope prompt asking you to indicate, for each of the problems, the page in which the solution to that problem is found. If you have solutions to more than one problem in a page, make sure that the problem number is large and clear.**

1. Let V be a vector space. Consider linearly independent vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in a subspace H of V and vectors $\vec{w}_1, \vec{w}_2, \ldots \vec{w}_q$ that span H. Show that there is a basis of H that consists of *all* the $\vec{v}_i$ and *some* of the $\vec{w}_j$. Hint: Set $\vec{v}_{p+k} = \vec{w}_k$, $1 \leq k \leq q$. If the set of $p+q$ vectors $\{\vec{v}_1, \ldots, \vec{v_{p+q}}\}$ is linearly dependent, then remove the first vector $\vec{v}_j$ among them, which is a linear combination of the preceding vectors $\{\vec{v}_1, \ldots \vec{v}_{j-1}\}$. Then repeat. Prove that you end up with a basis of H with the desired properties. You will need to use the Spanning Set Theorem 5 in section 4.3.

2. Let G and H be subspaces of a vector space V.
   (a) The intersection $G \cap H$ is the subset of V consisting of vectors that belong to both G and H. Show that $G \cap H$ is a subspace of V.
   (b) The sum $G + H$ is the subset of V consisting of sums of vectors $\vec{g} + \vec{h}$, where $\vec{g}$ belongs to G and $\vec{h}$ belongs to H. Show that $G + H$ is a subspace of V.
   (c) Let $\{v_1, \ldots, v_p\}$ be a linearly independent subset of V and k an integer satisfying $1 \leq k < p$. Show that $\text{span}\{v_1, v_2, \ldots, v_k\} \cap \text{span}\{v_{k+1}, \ldots, v_{n-1}, v_p\}$ is the zero subspace of V.
   (d) Let $A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Find a basis for $Null(A)$, $Col(A)$, $Null(A) \cap Col(A)$, and $Null(A) + Col(A)$.

3. Let G and H be finite dimensional subspaces of a vector space V. Show that

$$\dim(G) + \dim(H) = \dim(G \cap H) + \dim(G + H).$$

Hint: Let $\vec{v}_1, \ldots \vec{v}_m$ be a basis of $G \cap H$. Using problem 1 extend it to a basis $\vec{v}_1, \ldots \vec{v}_m, \vec{g}_1, \ldots \vec{g}_p$ of G and a basis $\vec{v}_1, \ldots, \vec{v}_m, \vec{h}_1, \ldots, \vec{h}_q$ of H. Show that

$$\vec{v}_1, \ldots, \vec{v}_m, \vec{g}_1, \ldots, \vec{g}_p, \vec{h}_1, \ldots, \vec{h}_q$$

is a basis of $G + H$. Hint: For the proof of linear independence, assume that $a_1\vec{v}_1 + \ldots + a_m\vec{v}_m + b_1\vec{g}_1 + \ldots + b_p\vec{g}_p + c_1\vec{h}_1 + \ldots + c_q\vec{h}_q = 0$. Show first that $c_1\vec{h}_1 + \ldots + c_q\vec{h}_q$ belongs to $G \cap H$ and use it to prove that $b_j = 0$, $1 \leq j \leq p$.

4. If G and H are subspaces of $\mathbb{R}^{10}$, with $\dim(G) = 6$ and $\dim(H) = 7$, what are the possible dimensions of $G \cap H$? Hint: Use problem 3.

5. Let $T : V \to W$ be a linear transformation, $\mathcal{B} = \{v_1, \ldots, v_n\}$ a basis for V, and $\mathcal{C} = \{w_1, \ldots, w_m\}$ a basis for W. Let $[\ ]_{\mathcal{C}} : W \to \mathbb{R}^m$ be the linear coordinate transformation of Theorem 8 in Section 4.4 of our textbook. Let $S : \mathbb{R}^n \to V$ be the linear transformation sending the vector $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ to $c_1v_1 + c_2v_2 + \ldots + c_nv_n$.

   Note that S is the inverse of $[\ ]_{\mathcal{B}}$.

   (a) By definition, the composite linear transformation $[\ ]_{\mathcal{C}} \circ T \circ S : \mathbb{R}^n \to \mathbb{R}^m$ maps $\vec{x}$ to $[T(S(\vec{x}))]_{\mathcal{C}}$. It fits in the diagram:
   $$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "{[\ ]_{\mathcal{C}}}"] \\ \mathbb{R}^n \arrow[u, "S"] \arrow[r, "{[\ ]_{\mathcal{C}} \circ T \circ S}"'] & \mathbb{R}^m. \end{tikzcd}$$
   Show that the standard matrix of $[\ ]_{\mathcal{C}} \circ T \circ S : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix

   $$A = ([T(v_1)]_{\mathcal{C}}[T(v_2)]_{\mathcal{C}} \ldots [T(v_n)]_{\mathcal{C}}),$$

   which j-th column is $[T(v_j)]_{\mathcal{C}}$. Hint: Compute A column by column. Note: The matrix A is considered in Theorem 15 of Section 4.7 of our textbook only in the special case where $V = W$. When $V = W$ and $\mathcal{B} = \mathcal{C}$ we will call A the *$\mathcal{B}$-matrix* of T.

   (b) Show that the coordinate linear transformation $[\ ]_{\mathcal{B}}$ maps $\ker(T)$ onto $Null(A)$. In other words, show that if $T(v) = 0$, then $[v]_{\mathcal{B}}$ belongs to $Null(A)$ and if $A\vec{x} = \vec{0}$, then $\vec{x} = [v]_{\mathcal{B}}$, for some vector v in $\ker(T)$. Conclude that $\dim(\ker(T)) = \dim(Null(A))$. Hint: Note the equality $A([v]_{\mathcal{B}}) = [T(v)]_{\mathcal{C}}$, for all v in V.

(c) Show that $[\ ]_{\mathcal{C}}$ maps the image $Im(T)$ onto $Col(A)$. Conclude that $\dim(Im(T)) = \dim(Col(A))$.

(d) Generalize the Rank-Nullity Theorem by proving the following equality $\dim(Im(T)) + \dim(\ker(T)) = \dim(V)$.

6. Let $M_{2\times 2}$ be the vector space of 2×2 matrices (with the usual scalar multiplication and addition of matrices). Consider the following basis of $M_{2\times 2}$.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let $C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Let $T : M_{2\times 2} \to M_{2\times 2}$ be given by $T(X) = XC - CX$.

(a) Show that T is a linear transformation.

(b) Compute the $\mathcal{B}$-matrix of T (as in Problem 5, but with $V = W = M_{2\times 2}$ and take $\mathcal{C}$ to be equal to the specific basis $\mathcal{B}$ given above).

(c) Use part 5b to find a basis for $\ker(T)$. (It should consist of 2×2 matrices, not vectors in $\mathbb{R}^4$).

(d) Use part 5c to find a basis of $Im(T)$.