Math 235H Written homework assignment 5

1. Let V be a vector space. Consider linearly independent vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in a subspace H of V and vectors $\vec{w}_1, \vec{w}_2, \ldots \vec{w}_q$ that span H. Show that there is a basis of H that consists of *all* the $\vec{v}_i$ and *some* of the $\vec{w}_j$. Hint: Use the Spanning Set Theorem 5 in section 4.3.

Consider the ordered set $\mathcal{S} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{p+q}\}$, where $\vec{u}_i = \vec{v}_i$, if $1 \le i \le p$, and $\vec{u}_{p+j} = \vec{w}_j$, $1 \le j \le q$.
If the set $\{\vec{u}_1, \ldots, \vec{u}_{p+q}\}$ is not linearly independent, then one $\vec{u}_K$ is a linear combination of the preceeding vectors $\vec{u}_1, \ldots, \vec{u}_{K-1}$, for some $K > p$ (since $\vec{u}_1, \ldots, \vec{u}_p$ is linearly independent. In that case we call $\vec{u}_K$ a redundent vector and $H = \operatorname{Span} \mathcal{S} = \operatorname{Span} \mathcal{S} \setminus \{\vec{u}_K\}$. Repeating

Since $\mathcal{S}$ contains $\{v_1, \ldots, v_p\}$

by the Spanning Set Theorem

the process with $S \setminus \{\vec{u}_K\}$ until all redundent vectors were discarded we arrive at a linearly independent set $\{\vec{u}_1, \ldots, \vec{u}_p, \vec{u}_{i_1}, \ldots, \vec{u}_{i_m}\}$, with $p < i_t \le p+q$ which spans H. It is thus a basis for H of the desired property, since $u_{i_t} = w_{(i_t - p)}$.

2. Let G and H be subspaces of a vector space V.

(a) The intersection $G \cap H$ is the subset of V consisting of vectors that belong to both G and H. Show that $G \cap H$ is a subspace of V.

(1) $G \cap H$ contains $\vec{0}$, since both G and H do being subspaces.
(2) Let $\vec{u}, \vec{v}$ be vectors in $G \cap H$, Then $\vec{u}+\vec{v}$ belong to G, since G is a subspace, and to H, for the same reason, hence to $G \cap H$.
(3) Let $\vec{u}$, be a vector in $G \cap H$, and c a scalar. Then $c\vec{u}$ belong to G, since G is a subspace, and to H, for the same reason, hence to $G \cap H$.

(b) The sum $G + H$ is the subset of V consisting of sums of vectors $\vec{g} + \vec{h}$, where $\vec{g}$ belongs to G and $\vec{h}$ belongs to H. Show that $G + H$ is a subspace of V.

(1) $\vec{0} = \vec{0} + \vec{0}$, and $\vec{0}$ belongs to G and to H. Hence $\vec{0}$ belongs to $G+H$.
(2) Let $\vec{u}, \vec{v}$ be vectors in $G+H$. Then $\vec{u} = \vec{g}_1 + \vec{h}_1$ and $\vec{v} = \vec{g}_2 + \vec{h}_2$, for $\vec{g}_i$ in G and $\vec{h}_i$ in H, $i=1,2$. So $\vec{u}+\vec{v} = (\vec{g}_1+\vec{g}_2)+(\vec{h}_1+\vec{h}_2)$ and $\vec{g}_1+\vec{g}_2$ belongs to G, since G is a subspace, and $\vec{h}_1+\vec{h}_2$ belongs to H for the same reason. Hence $\vec{u}+\vec{v}$ belongs to $G+H$.
(3) Let $\vec{u} = \vec{g}+\vec{h}$ be a vector in $G+H$, with $\vec{g}$ in G and $\vec{h}$ in H, and let c be a scalar. Then $c\vec{g}$ belongs to G, $c\vec{h}$ belongs to H, since G and H are subspaces. Thus $c\vec{u} = c\vec{g} + c\vec{h}$ belongs to $G+H$.

(c) Let $\{v_1, \ldots, v_p\}$ be a linearly independent subset of V and k an integer satisfying $1 \le k < p$. Show that $\underbrace{\operatorname{span}\{v_1, v_2, \ldots, v_k\}}_{G} \cap \underbrace{\operatorname{span}\{v_{k+1}, \ldots, v_{p-1}, v_p\}}_{H}$ is the zero subspace of V.

We already know that $G \cap H$ is a subspace, by Part (b). Suppose $\vec{v}$ is in $G \cap H$. Then $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ and $\vec{v} = c_{k+1}\vec{v}_{k+1} + \cdots + c_p\vec{v}_p$. So $\vec{0} = \vec{v} - \vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k - c_{k+1}\vec{v}_{k+1} - \cdots - c_p\vec{v}_p$.

Hence, $c_i = 0$, for $1 \le i \le p$, since $\{v_1, \ldots, v_p\}$ is linearly independent. So $\vec{v} = \vec{0}$.

x_1, x_4 free

(d) Let $A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$. Find a basis for $Null(A)$, $Col(A)$, $Null(A) \cap Col(A)$, and $Null(A) + Col(A)$.

<u>Null (A):</u> $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ -x_4 \\ x_4 \end{pmatrix} = x_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + x_4 \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}}_{v_2}$

$\{v_1, v_2\}$ is a basis for Null (A).

The pivot columns (2nd and third) of A are a basis for Col (A), namely $\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_3} \right\}$

<u>Null (A) ∩ Col (A):</u> If $\vec{v}$ is in Null (A) ∩ Col (A), then $\vec{v} = a_1 v_1 + a_2 v_2$ and $\vec{v} = b_1 v_1 + b_3 v_3$,
So $\vec{0} = (a_1 - b_1)\vec{v}_1 + a_2 \vec{v}_2 + b_3 \vec{v}_3$. But the set $\{v_1, v_2, v_3\}$ is linearly independent, so $a_1 = b_1$, $a_2 = 0$, and $b_3 = 0$, and so $\vec{v} = a_1 v_1$ belongs to span $\{v_1\}$. So $\{v_1\}$ is a basis for Null (A) ∩ Col (A).

<u>Null (A) + Col (A):</u> Let $\vec{v} = \vec{u} + \vec{w}$, where $\vec{u}$ is in Null (A) and $\vec{w}$ is in Col (A). Then $\vec{u} =$ 3 $a_1 v_1 + a_2 v_2$ and $\vec{w} = b_1 v_1 + b_3 v_3$.
So $\vec{v} = (a_1 + b_1)\vec{v}_1 + a_2 \vec{v}_2 + b_3 \vec{v}_3$. So $\vec{v}$ belongs to Span $\{v_1, v_2, v_3\}$.
Conversely, if $\vec{v}$ belongs to Span $\{v_1, v_2, v_3\}$ then $\vec{v} = (c_1 v_1 + c_2 v_2) + c_3 v_3$, with $(c_1 v_1 + c_2 v_2)$ in Null (A) and $c_3 v_3$ in Col (A) so $\vec{v}$ is in Null (A) + Col (A). Thus, Null (A) + Col (A) = Span $\{v_1, v_2, v_3\}$. Since $\{v_1, v_2, v_3\}$ is linearly independent, it is a basis for Null (A) + Col (A).

3. Let G and H be finite dimensional subspaces of a vector space V. Show that

(†) $$\dim(G) + \dim(H) = \dim(G \cap H) + \dim(G + H).$$

Hint: Let $\vec{v}_1, \ldots \vec{v}_m$ be a basis of $G \cap H$. Using problem 1 extend it to a basis $\vec{v}_1, \ldots \vec{v}_m, \vec{g}_1, \ldots \vec{g}_p$ of G and a basis $\vec{v}_1, \ldots, \vec{v}_m, \vec{h}_1, \ldots, \vec{h}_q$ of H. Show that

(*) $$\vec{v}_1, \ldots, \vec{v}_m, \vec{g}_1, \ldots, \vec{g}_p, \vec{h}_1, \ldots, \vec{h}_q$$

is a basis of $G + H$. Hint: For the proof of linear independence, assume that $a_1\vec{v}_1 + \ldots + a_m\vec{v}_m + b_1\vec{g}_1 + \ldots + b_p\vec{g}_p + c_1\vec{h}_1 + \ldots + c_q\vec{h}_q \overset{(**)}{=} 0$. Show first that $c_1\vec{h}_1 + \ldots + c_q\vec{h}_q$ belongs to $G \cap H$ and use it to prove that $b_j = 0$, $1 \le j \le p$.

If (*) is a basis for $G+H$, then $\dim(G+H) = m+p+q$
$\dim(G) = m+p$
$\dim(H) = m+q$
$\dim(G\cap H) = m$, so Eq (†) holds.

Proof that the set (*) is linearly independent:

Assume (**). Then $c_1\vec{h}_1 + \ldots + c_q\vec{h}$ both belongs to H and equal to $-(a_1\vec{v}_1 + \ldots + a_m\vec{v}_m + b_1\vec{g}_1 + \ldots + b_p\vec{g}_p)$ which belongs to G. Hence, $c_1\vec{h}_1 + \ldots + c_q\vec{h}_q$ belongs to $G\cap H$ and is thus a linear comb $d_1\vec{v}_1 + \ldots + d_m\vec{v}_m$ of its basis. So,
$\vec{0}$ = left hand side of (**) $= (a_1+d_1)\vec{v}_1 + \ldots + (a_m+d_m)\vec{v}_m + b_1\vec{g}_1 + \ldots + b_p\vec{g}_p$.
Hence, $b_i = 0$, for $1 \le i \le p$ (since $\vec{v}_1, \ldots, \vec{v}_m, \vec{g}_1, \ldots, \vec{g}_p$ is linearly indep.). Interchanging the roles of G and H we get that $c_i = 0$, for $1 \le i \le q$.
Hence, $\vec{0} = a_1\vec{v}_1 + \ldots + a_m\vec{v}_m$. So $a_i = 0$, for $0 \le i \le m$, since $\{v_1, \ldots, v_m\}$ is linearly independent.

Proof that (*) spans: Let $v \in G+H$. Then $v = g+h$, where g is in G and h is in H. So $g = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{p} b_i g_i$ and $h = \sum_{i=1}^{m} c_i v_i + \sum_{i=1}^{q} d_i h_i$.

Thus $v = \sum_{i=1}^{m} (a_i + c_i) v_i + \sum_{i=1}^{p} b_i g_i + \sum_{i=1}^{q} d_i h_i$ is a linear combination of the vectors in (*). $\square$

4. If G and H are subspaces of $\mathbb{R}^{10}$, with $\dim(G) = 6$ and $\dim(H) = 7$, what are the possible dimensions of $G \cap H$? Hint: Use problem 3.

$$\underbrace{\underset{6}{\overset{\|}{\dim(G)}} + \underset{7}{\overset{\|}{\dim(H)}}}_{13} = \dim(G \cap H) + \underbrace{\dim(G+H)}_{\leq \dim(\mathbb{R}^{10}) = 10}$$

So $\dim(G \cap H) \geq 13 - 10 = 3$.

$G \cap H$ is a subspace of G, so $\dim(G \cap H) \leq 6$.

We get that $3 \leq \dim(G \cap H) \leq 6$. We show next that all four possibilities are possible:

Let $\{e_1, \dots, e_{10}\}$ be the standard basis of $\mathbb{R}^{10}$, and let $H = \operatorname{Span}\{e_1, \dots, e_7\}$.

If $G = \operatorname{Span}\{e_1, \dots, e_6\}$ then $G \cap H = G$ and $\dim(G \cap H) = 6$.

If $G = \operatorname{Span}\{e_1, e_2, e_3, e_4, e_5, e_8\}$, then $G \cap H = \operatorname{Span}\{e_1, e_2, e_3, e_4, e_5\}$ and $\dim(G \cap H) = 5$.

If $G = \operatorname{Span}\{e_1, e_2, e_3, e_4, e_8, e_9\}$, then $G \cap H = \operatorname{Span}\{e_1, e_2, e_3, e_4\}$ and $\dim(G \cap H) = 4$.

If $G = \operatorname{Span}\{e_1, e_2, e_3, e_8, e_9, e_{10}\}$, then $G \cap H = \operatorname{Span}\{e_1, e_2, e_3\}$ and $\dim(G \cap H) = 3$.

Q.E.D

5 Let $T : V \to W$ be a linear transformation, $\mathcal{B} = \{v_1, \ldots, v_n\}$ a basis for V, and $\mathcal{C} = \{w_1, \ldots, w_m\}$ a basis for W. Let $[\ ]_\mathcal{C} : W \to \mathbb{R}^m$ be the linear coordinate transformation of Theorem 8 in Section 4.4 of our textbook. Let $S : \mathbb{R}^n \to V$ be the linear transformation sending the vector $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ to $c_1v_1 + c_2v_2 + \ldots + c_nv_n$.

Note that S is the inverse of $[\ ]_\mathcal{B}$.

(a) By definition, the composite linear transformation $[\ ]_\mathcal{C} \circ T \circ S : \mathbb{R}^n \to \mathbb{R}^m$ maps $\vec{x}$ to $[T(S(\vec{x}))]_\mathcal{C}$. It fits in the diagram:

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "{[\ ]_\mathcal{C}}"] \\ \mathbb{R}^n \arrow[u, "S"] \arrow[r, "{[\ ]_\mathcal{C} \circ T \circ S}"'] & \mathbb{R}^m. \end{tikzcd}$$

Show that the standard matrix of $[\ ]_\mathcal{C} \circ T \circ S : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix

$$A = ([T(v_1)]_\mathcal{C}[T(v_2)]_\mathcal{C} \ldots [T(v_n)]_\mathcal{C}),$$

which j-th column is $[T(v_j)]_\mathcal{C}$. Hint: Compute A column by column. Note: The matrix A is considered in Theorem 15 of Section 4.7 of our textbook only in the special case where $V = W$. When $V = W$ and $\mathcal{B} = \mathcal{C}$ we will call A the *$\mathcal{B}$-matrix* of T.

Write $A = (\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n)$ where $\vec{a}_j$ is the j-th column of A. We know that $A\vec{x} = [T(S(\vec{x}))]_\mathcal{C}$, for every $\vec{x}$ in $\mathbb{R}^n$, by definition of A. Let $\vec{e}_j$ be the j-th column of I, the $n \times n$ identity matrix.

$$\vec{a}_j = A\vec{e}_j = [T(\underbrace{S(\vec{e}_j)}_{\vec{v}_j})]_\mathcal{C} = [T(\vec{v}_j)]_\mathcal{C},$$

(b) Show that the coordinate linear transformation $[\ ]_{\mathcal{B}}$ maps $\ker(T)$ onto $Null(A)$. In other words, show that if $T(v) = 0$, then $[v]_{\mathcal{B}}$ belongs to $Null(A)$ and if $A\vec{x} = \vec{0}$, then $\vec{x} = [v]_{\mathcal{B}}$, for some vector v in $\ker(T)$. Conclude that $\dim(\ker(T)) = \dim(Null(A))$.

The equation $A\vec{x} = [T(S(\vec{x}))]_e$ for all $\vec{x}$ in $\mathbb{R}^n$ implies that $\boxed{A[\vec{v}]_{\mathcal{B}} \overset{(+)}{=} [T(\vec{v})]_e}$, for all $\vec{v}$ in V.
Indeed, take $\vec{x} = [\vec{v}]_{\mathcal{B}}$ and use the fact that $S([\vec{v}]_{\mathcal{B}}) = \vec{v}$.

If $\vec{v}$ is in $\ker(T)$, then $T(\vec{v}) = \vec{0}$, so $[T(\vec{v})]_e = \vec{0}$, so $A[\vec{v}]_{\mathcal{B}} = \vec{0}$, by equation (+), so $[\vec{v}]_{\mathcal{B}}$ is in $Null(A)$.

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be in $Null(A)$, and set $\vec{v} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$, so that $[\vec{v}]_{\mathcal{B}} = \vec{x}$.

Then $[T(\vec{v})]_e \overset{\text{by } (+)}{=} A[\vec{v}]_{\mathcal{B}} = A\vec{x} = \vec{0}$. Hence, the e-coordinate vector of $T(\vec{v})$ is zero. So $T(\vec{v}) = \vec{0}$, So $\vec{v}$ is in $\ker(T)$ and $\vec{x} = [v]_{\mathcal{B}}$. Hence $[\ ]_{\mathcal{B}}$ maps $\ker(T)$ __ONTO__ $Null(A)$.

Let $F : \ker(T) \to Null(A)$ be given by $F(\vec{v}) = [\vec{v}]_{\mathcal{B}}$. Then F is a linear transformation, since $[\ ]_{\mathcal{B}}$ is, F is one-to-one, since $[\ ]_{\mathcal{B}}$ is, and F is onto, as was shown above.

So F translates every linear algebra statement on $\ker(T)$ to one on $Null(A)$ (see the paragraph in Sec 4.4 before Example 5 in the text). In particular, F maps a linearly independent set to a linearly independent set.
Furthermore, F maps a set that spans $\ker(T)$ to a set that spans $Null(A)$. So if $\{u_1, \ldots, u_k\}$ is a basis for $\ker(T)$, then $\{[u_1]_{\mathcal{B}}, \ldots, [u_k]_{\mathcal{B}}\}$ is a basis for $Null(A)$. Thus $\dim(\ker(T)) = \dim(Null(A)) = k$.

$$\begin{array}{ccccc} V & \xrightarrow{T} & W & \supset & Im(T) \\ {[\ ]_B}\downarrow & & \downarrow [\ ]_C & & \\ \mathbb{R}^n & \xrightarrow[A]{} & \mathbb{R}^m & \supset & Col(A) \end{array}$$

(c) Show that $[\ ]_C$ maps the image $Im(T)$ onto $Col(A)$. Conclude that $\dim(Im(T)) = \dim(Col(A))$.

Let $\vec{w}$ be a vector in $Im(T) = \{T(v)$ such that $\vec{v}$ is in $V\}$. Then $\vec{w} = T(\vec{v})$ for some $\vec{v}$ in V. So

$[w]_C = [T(\vec{v})]_C = A[v]_B$. Now $A\vec{x}$ is in $Col(A)$ for every $\vec{x}$ in $\mathbb{R}^n$, Hence $[w]_C$ is in $Col(A)$.

Conversely, let $\vec{y}$ be a vector in $Col(A)$, $\vec{y} = \sum_{j=1}^{n} x_j \vec{a}_j$

Then $\vec{y} = A\vec{x}$, for $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$. So

$$\vec{y} = A\left[\sum_{j=1}^{n} x_j \vec{v}_j\right]_B = \left[T(\Sigma x_j \vec{v}_j)\right]_C, \text{ so}$$

$$\vec{y} = [\vec{w}]_C \quad \text{where } \vec{w} = T\left(\sum_{j=1}^{n} x_j \vec{v}_j\right) \text{ is in } Im(T),$$

Hence $[\ ]_C$ maps $Im(T)$ onto $Col(A)$.

Let $F : Im(T) \to Col(A)$ be given by $F(\vec{y}) = [\vec{y}]_C$. Then F is a one-to-one and onto linear transformation. Hence, $\dim Im(T) = \dim Col(A)$, by the argument in part b.

(d) Generalize the Rank-Nullity Theorem by proving the following equality $\dim(Im(T)) + \dim(\ker(T)) = \dim(V)$.

$\underbrace{\dim(Im(T))}_{\boxed{\text{Part c}}\ =\ \dim Col(A)} + \underbrace{\dim(\ker(T))}_{\boxed{\text{Part b}}\ =\ \dim(Null(A))}$

$\dim(Im(T)) + \dim(\ker(T)) = \dim(Col(A)) + \dim(Null(A)) =$ by the Rank-Nullity Theorem

$= n = \dim(V)$.

6 Let $M_{2\times 2}$ be the vector space of 2×2 matrices (with the usual scalar multiplication and addition of matrices). Consider the following basis of $M_{2\times 2}$.

$$\mathcal{B}=\left\{\underbrace{\begin{pmatrix}1&0\\0&0\end{pmatrix}}_{E_{11}},\underbrace{\begin{pmatrix}0&1\\0&0\end{pmatrix}}_{E_{12}},\underbrace{\begin{pmatrix}0&0\\1&0\end{pmatrix}}_{E_{21}},\underbrace{\begin{pmatrix}0&0\\0&1\end{pmatrix}}_{E_{22}}\right\}.$$

Let $C=\begin{pmatrix}1&1\\1&2\end{pmatrix}$. Let $T: M_{2\times 2}\to M_{2\times 2}$ be given by $T(B)=BC-CB$.

(a) Show that T is a linear transformation. (1) Let B, M be 2×2 matrices

$$T(B+M) = (B+M)C - C(B+M) = (BC-CB)+(MC-CM) = T(B)+T(M).$$

(2) Let λ be a scalar and B a 2×2 matrix, Then

$$T(\lambda B) = (\lambda B)C - C(\lambda B) = \lambda[BC-CB] = \lambda T(B).$$

(b) Compute the $\mathcal{B}$-matrix of T (as in Problem 8, but with $V=W=M_{2\times 2}$ and take $\mathcal{C}$ to be equal to the specific basis $\mathcal{B}$ given above).

$$\begin{array}{ccc} M_{2\times 2} & \xrightarrow{T} & M_{2\times 2} \\ \downarrow{[\ ]_{\mathcal{B}}} & & \downarrow{[\ ]_{\mathcal{B}}} \\ \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \end{array}$$

Mult by A the standard matrix of T.

We will show that

$$A=\begin{pmatrix}0&1&-1&0\\1&1&0&-1\\-1&0&-1&1\\0&-1&1&0\end{pmatrix}$$

$\boxed{\text{Part } 8a}$

$$\vec{a}_1 = [T(E_{11})]_{\mathcal{B}} = [\underbrace{E_{11}}_{\begin{pmatrix}1&0\\0&0\end{pmatrix}}\underbrace{C}_{\begin{pmatrix}1&1\\1&2\end{pmatrix}} - CE_{11}]_{\mathcal{B}} = [\underbrace{\begin{pmatrix}1&1\\0&0\end{pmatrix}-\begin{pmatrix}1&0\\1&0\end{pmatrix}}_{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}]_{\mathcal{B}} = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

$$\vec{a}_2 = [T(E_{12})]_{\mathcal{B}} = [\underbrace{E_{12}C}_{\begin{pmatrix}1&2\\0&0\end{pmatrix}} - \underbrace{CE_{12}}_{\begin{pmatrix}0&1\\0&1\end{pmatrix}}]_{\mathcal{B}} = \left[\begin{pmatrix}1&1\\0&-1\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}1\\1\\0\\-1\end{pmatrix}$$

$$\vec{a}_3 = [T(E_{21})]_{\mathcal{B}} = [\underbrace{E_{21}}_{\begin{pmatrix}0&0\\1&0\end{pmatrix}}\underbrace{C}_{\begin{pmatrix}1&1\\1&2\end{pmatrix}} - CE_{21}]_{\mathcal{B}} = [\underbrace{\begin{pmatrix}0&0\\1&1\end{pmatrix}-\begin{pmatrix}1&0\\2&0\end{pmatrix}}_{\begin{pmatrix}-1&0\\-1&1\end{pmatrix}}]_{\mathcal{B}} = \begin{pmatrix}-1\\0\\-1\\1\end{pmatrix}$$

$$\vec{a}_4 = [T(E_{22})]_{\mathcal{B}} = [\underbrace{E_{22}C - CE_{22}}_{\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1&1\\1&2\end{pmatrix}-\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}}]_{\mathcal{B}} = [\underbrace{\begin{pmatrix}0&0\\1&2\end{pmatrix}-\begin{pmatrix}0&1\\0&2\end{pmatrix}}_{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}]_{\mathcal{B}} = \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}.$$

Interchange R_1 and R_2

(c) Use part 8b to find a basis for $\ker(T)$. (It should consist of 2×2 matrices, not vectors in $\mathbb{R}^4$).

x_3 x_4 are free

$$A \sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \underset{\substack{\text{Add } R_2 \text{ to } R_4 \\ \text{Add } R_1 \text{ to } R_3}}{\sim} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\substack{\text{Add } -R_2 \text{ to } R_3 \\ \text{Add } -R_2 \text{ to } R_1}}{\sim} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3+x_4 \\ x_3 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{v_1} + x_4 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_2}$$

basis for Null(A).

$$v_1 = \Big[\underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}}_{C-2I}\Big]_B, \quad v_2 = \Big[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}\Big]_B.$$

So $\{C-2I, I\}$ is a basis for $\ker(T)$.

(d) Use part 8c to find a basis of $Im(T)$.

Basis for Col(A): The two pivot columns of A are a basis, $\vec{a}_1$ and $\vec{a}_2$

$$\vec{a}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]_B, \quad \vec{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\right]_B$$

So, $\Big\{ \underset{T(E_{11})}{\underset{\|}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}}, \underset{T(E_{12})}{\underset{\|}{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}}} \Big\}$ is a basis for $Im(T)$.