1 Let $A = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{pmatrix}$ be a 4×4 matrix with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. Compute det $\underbrace{\begin{pmatrix} 6\vec{v}_1 + 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{pmatrix}}_{B}$, if $\det(A) = 4$.

Let $C = \begin{pmatrix} \vec{0} \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{pmatrix}$ be the 4×4 matrix obtained from B by adding -2(Row 4 of B) to (Row 1 of B). Then the first row of C is $\vec{0}$ and $\det(B) = \det(C) = 0$.

2 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix A. Assume that for two non-zero vectors $\vec{v}_1$ and $\vec{v}_2$ we have $T(\vec{v}_1) = 5\vec{v}_1$ and $T(\vec{v}_2) = 7\vec{v}_2$.

(a) Show that v_1 and v_2 must be linearly independent. Proof by contradiction.

Assume that $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent. Then $\vec{v}_2 = c\vec{v}_1$, for some scalar c, since $\vec{v}_1 \neq \vec{0}$.

So $7\vec{v}_2 = T(\vec{v}_2) = T(c\vec{v}_1) = cT(\vec{v}_1) = c(5\vec{v}_1) = 5(c\vec{v}_1) = 5\vec{v}_2$.

So $2\vec{v}_2 = \vec{0}$. So $\vec{v}_2 = \vec{0}$. But we are given that $\vec{v}_2 \neq \vec{0}$. A contradiction.

(b) Show that $\det(A) = 35$. Hint: Let B be the matrix $(\vec{v}_1 \vec{v}_2)$ with columns $\vec{v}_1$ and $\vec{v}_2$. Compute the determinant of the product $\det(AB)$ in two ways.

Note that $AB = A(\vec{v}_1\,\vec{v}_2) = (A\vec{v}_1\; A\vec{v}_2) = (5\vec{v}_1\; 7\vec{v}_2)$.

So $\det(AB) = \det(\underbrace{(AB)^T}_{\text{transpose}}) = \underbrace{5\cdot 7}_{35}\det\begin{pmatrix}\vec{v}_1^{\,T}\\ \vec{v}_2^{\,T}\end{pmatrix} = 35\det(B)$,

On the other hand, $\det(AB) = \det(A)\cdot\det(B)$.

Hence $\det(A)\det(B) = 35\det(B)$.

Now $\det(B) \neq 0$, since B is invertible, by Part (a).

Hence, $\det(A) = 35$.

3 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of the plane with respect to a line L through the origin and A the standard matrix of T. Show that $\det(A) = -1$. Hint: Use parts 6(b) and 6(c) in Homework 2 and immitate the argument you used in problem 5.

b) Next compute the determinant of the standard matrix of the reflection R of $\mathbb{R}^3$ with respect to a plane given in Equation (1) (for any $\vec{u}$).

a) Let $\vec{v}_1$ be a non-zero vector in L and let $\vec{v}_2$ be a non zero vector in $\mathbb{R}^2$ orthogonal to L. Then $\vec{v}_1$ and $\vec{v}_2$ are linearly independent and so if $B = (\vec{v}_1\,\vec{v}_2)$ then $\det(B) \neq 0$.

Part 6(b) and 6(c) in HW 2 state: $T(\vec{v}_1) = \vec{v}_1$ and $T(v_2) = (-1)\vec{v}_2$.

So $\det(A)\det(B) = \det(AB) = \det(A(\vec{v}_1\,\vec{v}_2)) = \det(\vec{v}_1\; -\vec{v}_2) = -\det(v_1\, v_2) =$
$= -\det(B)$.

Hence, $\det(A) = -1$, since $\det(B) \neq 0$.

Let $R: \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in formula (1). Let $\vec{v}_1 = \vec{u}$. Let $\vec{v}_2, \vec{v}_3$ be two linearly independent vectors in the plane $\underline{P}$ orthogonal to L. Let B be the 3×3 matrix $(\vec{v}_1 \vec{v}_2 \vec{v}_3)$. Let A be the standard matrix of R. Then

$$\det(AB) \overset{*}{=} \det(A)\det(B)$$

Now $AB = (A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3) = (-\vec{v}_1\ \vec{v}_2\ \vec{v}_3)$,

$$\underbrace{A\vec{v}_1}_{\substack{\| \\ -\vec{v}_1 \\ \text{Problem 1(a)}}} \quad \underbrace{A\vec{v}_2}_{\substack{\| \\ \vec{v}_2}} \quad \underbrace{A\vec{v}_3}_{\substack{\| \\ \vec{v}_3 \\ \text{Problem 1(b)}}}$$

So, $\det(AB) = \det(\underbrace{(AB)^T}_{\text{transpose}}) = \det\begin{pmatrix} -v_1^T \\ v_2^T \\ v_3^T \end{pmatrix} = -\det\begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix} =$

$= \det(v_1 v_2 v_3) = -\det(B)$, we get

$$\det(AB) \overset{**}{=} -\det(B).$$

Combining $*$ and $**$ we get

$$\det(A)\det(B) = -\det(B).$$

Hence, $\det(A) = -1$, since $\det(B) \neq 0$ (the columns of B are linearly independent).

4

Let A be an $n \times n$ matrix, I_n the $n \times n$ identity matrix, and O the $n \times n$ matrix all of which entries are zero. Show that if $A^2 + 3A + 4I_n = O$, then A is invertible. Hint: $3A = 3I_nA$. Then express A^{-1} in terms of A.

$$\Leftrightarrow A^2 + 3I_m A + 4I_m = O$$

$$(A^2 + 3I_m)A = -4I_m$$

$$\Leftrightarrow -\tfrac{1}{4}(A + 3I_m)A = I_m.$$ Set $C = -\frac{1}{4}(A + 3I_m)$. Then $CA = I_m$. Hence A is invertible and $C = A^{-1}$, by part j of the Invertible Matrix Theorem in section 2.3 of our textbook.

5

Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{pmatrix}$ for two distinct numbers a and b. We define the function $f(t) = \det(A)$.

(a) Show that $f(t)$ is a quadratic polynomial in t. What is the coefficient of t^2.

$f(t) = \det(A) =$ cofactor expansion along 3-rd column

$$= 1 \underbrace{\det\begin{pmatrix} a & b \\ a^2 & b^2 \end{pmatrix}}_{ab(b-a)} - t \underbrace{\det\begin{pmatrix} 1 & 1 \\ a^2 & b^2 \end{pmatrix}}_{b^2-a^2} + t^2 \underbrace{\det\begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix}}_{b-a} = (b-a)t^2 + (a^2-b^2)t + ab(b-a).$$

The coefficient of t^2 is $b-a$.

(b) Explain why $f(a) = f(b) = 0$. Conclude that $f(t) = k(t-a)(t-b)$, for some constant k. Find the constant k using your work in part 8a.

$f(a) = \det\begin{pmatrix} 1 & 1 & 1 \\ a & b & a \\ a^2 & b^2 & a^2 \end{pmatrix} = 0$, since the first and third columns of A are equal, so the columns are NOT ~~linearly~~ independent and A is NOT invertible. Similarly, $f(b) = \det\begin{pmatrix} 1 & 1 & 1 \\ a & b & b \\ a^2 & b^2 & b^2 \end{pmatrix} = 0$.

So $f(t) = \underbrace{(b-a)}_{\text{coefficient of } t^2 \text{ from part (a).}}(t-a)(t-b)$

(c) For which values of t is the matrix A invertible?

We are given that $a \neq b$. Hence, $(b-a) \neq 0$ and $f(t)$ vanishes only for $t = a$ or $t = b$. Hence, A is invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow (t \neq a$ and $t \neq b)$.