Math 235H Written homework assignment 3 Fall 2020

• **Please follow carefully the instructions on the document titled "Submitting PDF documents via Gradescope" (see the PDF file attachment to the announcement "Gradescope - our hand written homework submission platform" on our Moodle page). After you upload the PDF file to Gradescope you will need also to respond to the Gradescope prompt asking you to indicate, for each of the problems, the page in which the solution to that problem is found. If you have solutions to more than one problem in a page, make sure that the problem number is large and clear.**

1. (a) Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Suppose that $AB = I_m$ (the $m \times m$ identity matrix). Show that the equation $A\vec{x} = \vec{b}$ is consistent, for every vector $\vec{b}$ in $\mathbb{R}^m$ and the equation $B\vec{x} = \vec{0}$ has only the trivial solution.

   (b) Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^n$ be linear transformations. Suppose that $S(T(\vec{x})) = \vec{x}$, for every vector $\vec{x}$ in $\mathbb{R}^m$. Show that T is one-to-one and S is onto.

2. Let L be the line in $\mathbb{R}^3$ spanned by a non-zero vector $\vec{u}$. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the formula
$$R(\vec{x}) \;=\; \vec{x} - 2\left(\frac{\vec{u}^T\vec{x}}{\vec{u}^T\vec{u}}\right)\vec{u}. \tag{1}$$
Note that vectors in $\mathbb{R}^3$ are considered here as 3×1 matrices, and we regard the 1×1 matrices $\vec{u}^T\vec{x}$ and $\vec{u}^T\vec{u}$ as scalars, so that the fraction $\frac{\vec{u}^T\vec{x}}{\vec{u}^T\vec{u}}$ is a scalar (quotient of two dot products).

   (a) Show that R is a linear transformation by verifying properties (1) and (2) in the definition of a linear transformation in Section 1.8. Hint: Note that $\vec{u}^T\vec{x}$ is a linear transformation $D : \mathbb{R}^3 \to \mathbb{R}^1$, by Theorem 5 in Section 1.4. This will enable you to argue more concisely, without referring to the coordinates of $\vec{x}$ and $\vec{u}$.

   (b) Show that $R(\vec{v}) = -\vec{v}$, for every vector $\vec{v}$ in L.

   (c) Two vectors $\vec{v}$, $\vec{w}$ in $\mathbb{R}^3$ are orthogonal (i.e., perpendicular), if $\vec{v}^T\vec{w} = 0$. Show that $R(\vec{x}) = \vec{x}$, if $\vec{x}$ belongs to the plane P orthogonal to L. *Note: Parts 2b and 2c show that R is the reflection of $\mathbb{R}^3$ with respect to the plane P.*

   (d) Find the standard matrix of the reflection $R : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the plane $x_1 + x_2 + x_3 = 0$. Hint: Let $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

3. Consider the matrix $D_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta & \cos(\theta) \end{pmatrix}$. We know that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by $T(\vec{x}) = D_\theta\vec{x}$ is a counterclockwise rotation of the plane with angle θ about the origin.

(a) For two angles α and β consider the two products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Arguing geometrically, describe the two linear transformations with standard matrices $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Are they the same?

(b) Now compute the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Do the results make sense in terms of your answer in part 3a? Recall the trig identities

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).\end{aligned}$$

4. Let P and Q be two lines through the origin in $\mathbb{R}^2$ with an angle $\pi/4 < \theta < \pi/2$ between them (the condition on the range of θ is not necessary, but will result in a nicer sketch in part 4a). Let $R_P : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of the plane about the line P, as in Homework 2 Question 5. Define R_Q similarly using the line Q. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be their composition, given by $T(\vec{x}) = R_Q(R_P(\vec{x}))$.

(a) Sketch a diagram assuming that the intersection of P and Q with the first quadrant are two rays not on the x_1 and x_2 axes, and that Q lies above P in the first quadrant. Then sketch $T\begin{pmatrix}1\\0\end{pmatrix}$. Express the angle α the vector $T\begin{pmatrix}1\\0\end{pmatrix}$ makes with the positive half of the x_1-axis in terms of θ and use it to express $T\begin{pmatrix}1\\0\end{pmatrix}$ in terms of θ.

(b) Note that for a sufficiently small positive scalar t, the vector $\vec{v} := \begin{pmatrix}1\\t\end{pmatrix}$ also lies in the first quadrant below the line P. Choose such a scalar t and sketch $\vec{v}$ and $T(\vec{v})$ in a diagram and use your sketch to conclude that the angle between v and $T(\vec{v})$ is equal to the angle α between $\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}1\\0\end{pmatrix}$ you computed in part 4a.

(c) Let S be the counterclockwise rotation of the plane angle α about the origin. Show that T and S must be the same linear transformation. I.e., prove the equality $T(\vec{x}) = S(\vec{x})$, for every vector $\vec{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ in $\mathbb{R}^2$. *Hint: Use the linearity of T and S.*

(d) Conclude that the composition T of the two reflections is a rotation and express the angle of rotation in terms of the angle θ between the two lines.