1 (a) Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Suppose that $AB = I_m$ (the $m \times m$ identity matrix). Show that the equation $A\vec{x} = \vec{b}$ is consistent, for every vector $\vec{b}$ in $\mathbb{R}^m$ and the equation $B\vec{x} = \vec{0}$ has only the trivial solution.

- Let $\vec{b}$ be a vector in $\mathbb{R}^m$. Then $\vec{x} = B\vec{b}$, is a solution of $A\vec{x} = \vec{b}$ because $A(B\vec{b}) = (AB)\vec{b} = I_m\vec{b} = \vec{b}$. Hence $A\vec{x} = \vec{b}$ is consistent.
- Assume $B\vec{x} = \vec{0}$. Then $A(B\vec{x}) = A\vec{0} = \vec{0}$. On the other hand, $A(B\vec{x}) = (AB)\vec{x} = I_m\vec{x} = \vec{x}$. So $\vec{x} = \vec{0}$. Hence, $B\vec{x} = \vec{0}$ has only the trivial solution.

(b) Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^n$ be linear transformations. Suppose that $S(T(\vec{x})) = \vec{x}$, for every vector $\vec{x}$ in $\mathbb{R}^m$. Show that T is one-to-one and S is onto.

Notice that part (a) is a translation of part (b) if we take A to be the standard matrix of S and B that of T.

<u>S is onto:</u> Let $\vec{b}$ be a vector in $\mathbb{R}^m$. Then $S(T(\vec{b})) = \vec{b}$. So $\vec{b}$ is a value of S.

<u>T is one-to-one:</u> Suppose $T(\vec{u}) = T(\vec{v})$, we need to show that $\vec{u} = \vec{v}$. Indeed, if $T(\vec{u}) = T(\vec{v})$, then

$$\underbrace{S(T(\vec{u}))}_{=\vec{u}} = \underbrace{S(T(\vec{v}))}_{=\vec{v}}$$

The left hand side is $\vec{u}$, the right " " " $\vec{v}$.

So $\vec{u} = \vec{v}$.

2. Let L be the line in $\mathbb{R}^3$ spanned by a non-zero vector $\vec{u}$. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the formula

$$R(\vec{x}) \;=\; \vec{x} - 2\left(\frac{\vec{x}^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u}. \qquad (1)$$

Note that vectors in $\mathbb{R}^3$ are considered here as 3×1 matrices, and we regard the 1×1 matrices $\vec{x}^T\vec{u}$ and $\vec{u}^T\vec{u}$ as scalars, so that the fraction $\frac{\vec{x}^T\vec{u}}{\vec{u}^T\vec{u}}$ is a scalar (quotient of two dot products).

(a) Show that R is a linear transformation by verifying properties (1) and (2) in the definition of a linear transformation in Section 1.8.

(1) $R(\vec{v}+\vec{w}) = (\vec{v}+\vec{w}) - 2\left(\frac{(v+w)^T\vec{u}}{u^Tu}\right)\vec{u} = \vec{v} - 2\left(\frac{v^Tu}{u^Tu}\right)\vec{u} + \vec{w} - 2\left(\frac{\vec{w}^T\vec{u}}{u^Tu}\right)\vec{u} = R(\vec{v}) + R(\vec{w}).$

The above holds for every two vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^3$

(2) For every vector $\vec{w}$ in $\mathbb{R}^3$ and any scalar c we have

$R(c\vec{w}) = c\vec{w} - 2\left(\frac{(c\vec{w})^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u} = c\left[\vec{w} - 2\left(\frac{\vec{w}^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u}\right] = cR(\vec{w}).$

(b) Show that $R(\vec{v}) = -\vec{v}$, for every vector $\vec{v}$ in L.

If $\vec{v}$ belongs to L, then $\vec{v} = c\vec{u}$, for some scalar c and

so $R(\vec{v}) = R(c\vec{u}) \underset{\substack{\uparrow \\ \text{Part a}}}{=} cR(\vec{u}) = c\left[\vec{u} - 2\underbrace{\left(\frac{\vec{u}^T\vec{u}}{\vec{u}^T\vec{u}}\right)}_{1}\vec{u}\right] = c\left[\vec{u} - 2\vec{u}\right] = -c\vec{u} = -\vec{v}.$

(c) Two vectors $\vec{v}$, $\vec{w}$ in $\mathbb{R}^3$ are orthogonal (i.e., perpendicular), if $\vec{v}^T\vec{w} = 0$. Show that $R(\vec{x}) = \vec{x}$, if $\vec{x}$ belongs to the plane P orthogonal to L. *Note: Parts 1b and 1c show that R is the reflection of $\mathbb{R}^3$ with respect to the plane P.*

If $\vec{x}$ belongs to P, then $\vec{x}^T\vec{u} = 0$, so

$R(\vec{x}) = \vec{x} - 2\left(\frac{\vec{x}^Tu}{u^Tu}\right)\vec{u} = \vec{x} - 2\cdot 0\vec{u} = \vec{x}.$

$A = (\vec{a}_1\ \vec{a}_2\ \vec{a}_3)$

(d) Find the standard matrix of the reflection $R : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the plane $\underset{P}{x_1 + x_2 + x_3 = 0}$. Hint: Let $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Note that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x_1 + x_2 + x_3$,

So the plane P is orthogonal to $\vec{u}$.

$\vec{a}_1 = A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = R\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2\left(\dfrac{(1\,0\,0)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{(1\,1\,1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \dfrac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ -2/3 \end{pmatrix}$

$\vec{a}_2 = A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = R\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \dfrac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$

$\vec{a}_3 = A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = R\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \dfrac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$.

So $A = \dfrac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$

3 Consider the matrix $D_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. We know that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by $T(\vec{x}) = D_\theta \vec{x}$ is a counterclockwise rotation of the plane with angle θ about the origin.

(a) For two angles α and β consider the two products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Arguing geometrically, describe the two linear transformations with standard matrices $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Are they the same?

$D_\alpha D_\beta(\vec{x}) = \vec{x}$ rotated angle β and then angle α counterclockwise
$= \vec{x}$ " " $\alpha+\beta$ counterclockwise $= D_{\alpha+\beta}(\vec{x})$,

$D_\beta D_\alpha(\vec{x}) = D_{\alpha+\beta}(\vec{x})$, by interchanging the roles of α and β

(b) Now compute the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Do the results make sense in terms of your answer in part 2a? Recall the trig identities

$$\sin(\alpha+\beta) \overset{(*)}{=} \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) \overset{(**)}{=} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

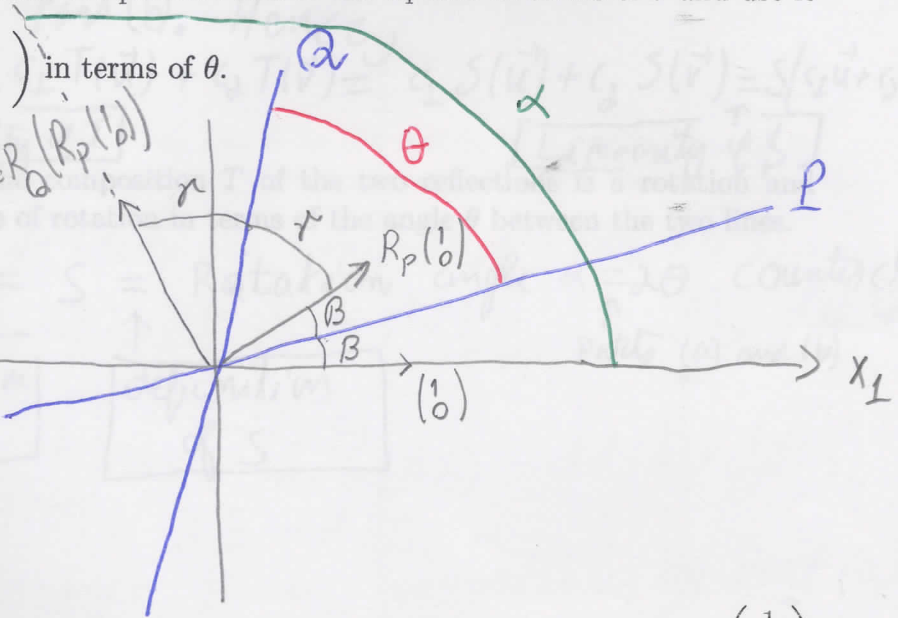
$D_\alpha D_\beta = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} = \begin{pmatrix} [\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] & -[\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)] \\ [\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)] & [-\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta)] \end{pmatrix}$

$\overset{(*) \text{ and } (**)}{=} \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = D_{\alpha+\beta}$

4

Let P and Q be two lines through the origin in $\mathbb{R}^2$ with an angle $\pi/4 < \theta < \pi/2$ between them. Let $R_P : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflections of the plane about the line P, as in Homework 2 Question 6. Define R_Q similarly using the line Q. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be their composition, given by $T(\vec{x}) = R_Q(R_P(\vec{x}))$.

(a) Sketch a diagram assuming that the intersection of P and Q with the first quadrant are two rays not on the x_1 and x_2 axes, and that Q lies above P in the first quadrant. Then sketch $T\begin{pmatrix}1\\0\end{pmatrix}$. Express the angle α the vector $T\begin{pmatrix}1\\0\end{pmatrix}$ makes with the positive half of the x_1-axis in terms of θ and use it to express $T\begin{pmatrix}1\\0\end{pmatrix}$ in terms of θ.



Angle $\theta = \beta + \gamma$

Angle $\alpha =$

$= 2\beta + 2\gamma$

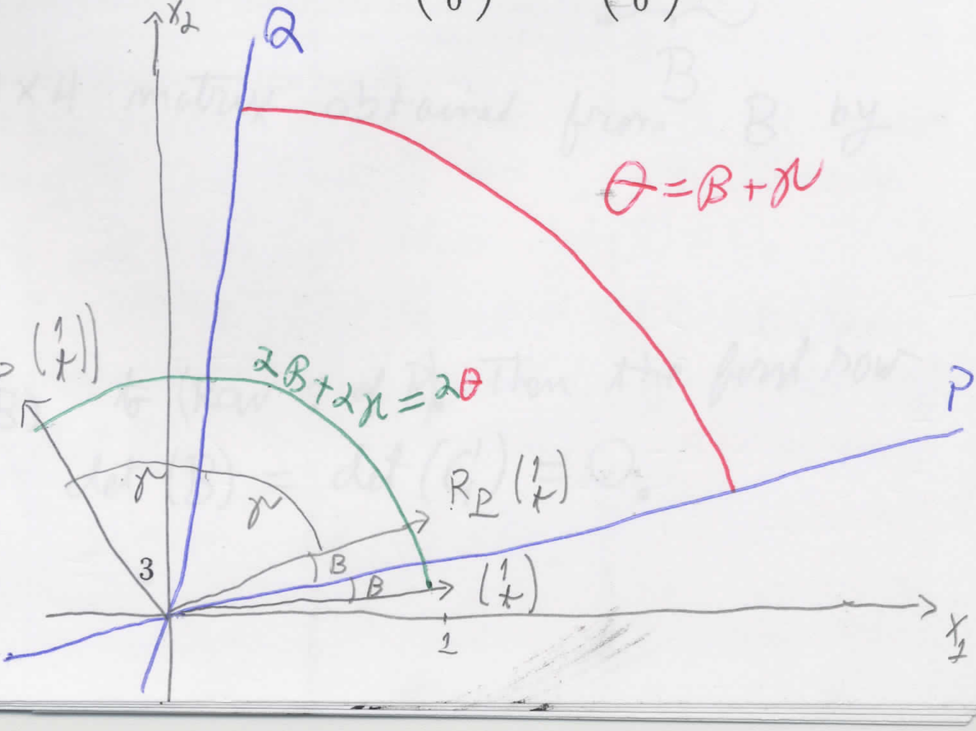
$= 2(\beta + \gamma) = 2\theta,$

so $\boxed{\alpha = 2\theta}$

(b) Note that for a sufficiently small positive scalar t, the vector $\vec{v} := \begin{pmatrix}1\\t\end{pmatrix}$ also lies in the first quadrant below the line P. Choose such a scalar t and sketch $\vec{v}$ and $T(\vec{v})$ in a diagram and use your sketch to conclude that the angle between v and $T(\vec{v})$ is equal to the angle α between $\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}1\\0\end{pmatrix}$ you computed in part 3a.

same argument shows

$T\left(\binom{1}{t}\right) = R_Q(R_P\binom{1}{t})$



(c) Let S be the counterclockwise rotation of the plane angle α about the origin. Show that T and S must be the same linear transformation. I.e., prove the equality $T(\vec{x}) = S(\vec{x})$, for every vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbb{R}^2$. *Hint: Use the linearity of T and S.*

The matrix $\begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix}$ has a pivot in every row, since $t \neq 0$. Hence, the two vectors $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ t \end{pmatrix}$ span $\mathbb{R}^2$. Every vector $\vec{x}$ in $\mathbb{R}^2$ is a linear combination $\vec{x} = c_1\vec{u} + c_2\vec{v}$. Now $T(\vec{u}) = S(\vec{u})$, by Part (a), and $T(\vec{v}) = S(\vec{v})$, by Part (b). Hence,

$$T(\vec{x}) = T(c_1\vec{u} + c_2\vec{v}) \underset{\boxed{\text{Linearity of } T}}{=} c_1 T(\vec{u}) + c_2 T(\vec{v}) = c_1 S(\vec{u}) + c_2 S(\vec{v}) \underset{\boxed{\text{Linearity of } S}}{=} S(c_1\vec{u} + c_2\vec{v}) = S(\vec{x})$$

(d) Conclude that the composition T of the two reflections is a rotation and express the angle of rotation in terms of the angle θ between the two lines.

Thus $R_Q \circ R_P \underset{\boxed{\text{definition of } T}}{=} T = S \underset{\boxed{\text{definition of } S}}{=}$ Rotation angle $\alpha \underset{\text{Parts (a) and (b)}}{=} 2\theta$ counterclockwise