Math 235H Written homework assignment 2 Fall 2020

• **Please follow carefully the instructions on the document titled "Submitting PDF documents via Gradescope" (see the PDF file attachment to the announcement "Gradescope - our hand written homework submission platform" on our Moodle page). After you upload the PDF file to Gradescope you will need also to respond to the Gradescope prompt asking you to indicate, for each of the problems, the page in which the solution to that problem is found. If you have solutions to more than one problem in a page, make sure that the problem number is large and clear.**

1. Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Show that their composition $U : \mathbb{R}^p \to \mathbb{R}^m$, given by $U(\vec{x}) = T(S(\vec{x})$, is a linear transformation by directly verifying the linearity properties (1) and (2) in the Definition of a linear transformation in Section 1.8.

2. Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^5$. Show that if the set $\{v_1, v_2, v_3\}$ is linearly independent and v_4 does not belong to span$\{v_1, v_2, v_3\}$, then the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent (use Theorem 7 in Section 1.7).

3. Let $\{v_1, v_2, v_3\}$ be a linearly independent set in $\mathbb{R}^n$. Show that the set $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is linearly independent as well.

4. (Supplementary Exercise 19 for Ch. 1) Recall that a line L in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ of the form $\{\vec{p} + t\vec{v} \ : \ t \text{ a real number}\}$, where $\vec{p}$ and $\vec{v}$ are fixed vectors and $\vec{v} \neq \vec{0}$. Suppose that $\{v_1, v_2, v_3\}$ are coordinate vectors of points on one line in $\mathbb{R}^3$ (which need not pass through the origin). Show that the set $\{v_1, v_2, v_3\}$ is linearly dependent.

5. Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be a vector in $\mathbb{R}^2$. Consider its transpose $\vec{u}^T = (u_1, u_2)$ as a 1×2 matrix so that for any vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbb{R}^2$ we get the 1×1 matrix $\vec{u}^T\vec{x} = (u_1x_1 + u_2x_2)$, which we identify with the scalar $u_1x_1 + u_2x_2$ in $\mathbb{R}$. The scalar $\vec{u}^T\vec{x}$ is called the *dot product* of $\vec{u}$ and $\vec{x}$. The transformation $D : \mathbb{R}^2 \to \mathbb{R}$, given by $D(\vec{x}) = \vec{u}^T\vec{x}$, is thus a linear transformation whose standard matrix is $\vec{u}^T$.

   Assume now that $\vec{u}$ is a **non-zero** vector in $\mathbb{R}^2$. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be given by the formula
   $$R(\vec{x}) \ = \ 2\left(\frac{\vec{u}^T\vec{x}}{\vec{u}^T\vec{u}}\right)\vec{u} - \vec{x}.$$
   Note that the fraction $\frac{\vec{u}^T\vec{x}}{\vec{u}^T\vec{u}}$ is a scalar (quotient of two dot products).

   (a) Show that R is a linear transformation by verifying properties (1) and (2) in the definition of a linear transformation in Section 1.8. Do this using the linearity of D and the equality $R(\vec{x}) = \left(\frac{2D(\vec{x})}{\vec{u}^T\vec{u}}\right)\vec{u} - \vec{x}$ avoiding reference to the coordinates of $\vec{u}$ and $\vec{x}$.

(b) Let L be the line $\text{span}\{\vec{u}\}$ in $\mathbb{R}^2$. Show that $R(\vec{v}) = \vec{v}$, for every vector $\vec{v}$ in L.

(c) Two vectors $\vec{v}$, $\vec{w}$ in $\mathbb{R}^2$ are orthogonal (i.e., perpendicular), if $\vec{v}^T\vec{w} = 0$. Show that $R(\vec{x}) = -\vec{x}$, if $\vec{x}$ is orthogonal to $\vec{u}$. *Note: Parts 5b and 5c show that R is the reflection of $\mathbb{R}^2$ with respect to the line L.*

(d) Let $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $L = \text{span}\{\vec{u}\}$ as above. Find the standard matrix of the reflection $R : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to L.