Math 235H Written homework assignment 2 Fall 2020

1. Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Show that their composition $U : \mathbb{R}^p \to \mathbb{R}^m$, given by $U(\vec{x}) = T(S(\vec{x}))$, is a linear transformation by directly verifying the linearity properties (1) and (2) in the Definition of a linear transformation in Section 1.8.

(1) Let v, w be vectors in $\mathbb{R}^P$. Then

$$U(v+w) = T(S(v+w)) \underset{\substack{\uparrow \\ \text{Linearity of } S}}{=} T(S(v)+S(w)) \underset{\substack{\uparrow \\ \text{Linearity of } T}}{=} T(S(v))+T(S(w)) = U(v)+U(w).$$

(2) Let $\vec{v}$ be a vector in $\mathbb{R}^P$ and c a scalar. Then

$$U(c\vec{v}) = T(S(c\vec{v})) \underset{\substack{\uparrow \\ \text{Linearity of } S}}{=} T(cS(\vec{v})) \underset{\substack{\uparrow \\ \text{Linearity of } T}}{=} cT(S(\vec{v})) = cU(\vec{v}).$$

2. Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^5$. Show that if the set $\{v_1, v_2, v_3\}$ is linearly independent and v_4 does not belong to $\text{span}\{v_1, v_2, v_3\}$, then the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent (use Theorem 7 in Section 1.7).

We need to show that if $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$, then $c_1 = c_2 = c_3 = c_4 = 0$. If $c_4 \neq 0$, then $\vec{v}_4 = \left(-\frac{c_1}{c_4}\right)\vec{v}_1 + \left(-\frac{c_2}{c_4}\right)\vec{v}_2 + \left(-\frac{c_3}{c_4}\right)\vec{v}_3$ so $\vec{v}_4$ would be a linear combination of v_1, v_2, v_3. But v_4 does not belong to $\text{span}\{v_1, v_2, v_3\}$. A contradiction. Hence, $c_4 = 0$. So $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$.

It follows that $c_1 = c_2 = c_3 = 0$, since $\{v_1, v_2, v_3\}$ is linearly independent. Thus, $c_i = 0$ for $1 \leq i \leq 4$.

Q.E.D

3. Let $\{v_1, v_2, v_3\}$ be a linearly independent set in $\mathbb{R}^n$. Show that the set $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is linearly independent as well.

Assume that $c_1 \vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) \overset{(*)}{=} \vec{0}$.

We need to show that $c_1 = c_2 = c_3 = 0$.

Now $(*)$ is equivalent to

$(c_1 + c_2 + c_3)\vec{v}_1 + (c_2 + c_3)\vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$. Hence

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}\right\}$$ since the set $\{v_1, v_2, v_3\}$ is linearly independent.

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

But the matrix A has a pivot in every column. Hence $c_1 = c_2 = c_3 = 0$. Q.E.D

4. (Supplementary Exercise 19 for Ch. 1) Recall that a line L in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ of the form $\{\vec{p} + t\vec{v} \ : \ t \text{ a real number}\}$, where $\vec{p}$ and $\vec{v}$ are fixed vectors and $\vec{v} \neq \vec{0}$. Suppose that $\{v_1, v_2, v_3\}$ are coordinate vectors of points on one line in $\mathbb{R}^3$ (which need not pass through the origin). Show that the set $\{v_1, v_2, v_3\}$ is linearly dependent.

$\vec{V}_2 - \vec{V}_1$ is a scalar multiple of $\vec{v}$, say $\vec{V}_2 - \vec{V}_1 = a\vec{v}$,

$\vec{V}_3 - \vec{V}_1$ " " " " " ", " $\vec{V}_3 - \vec{V}_1 = b\vec{v}$.

If $a = 0$, then $\vec{V}_2 = \vec{V}_1$, so $1\vec{V}_1 + (-1)\vec{V}_2 + 0\vec{V}_3 = 0$. $(*)$

If $a \neq 0$ and $b \neq 0$, then

$$b(V_2 - V_1) + a(V_3 - V_1) = ab\vec{v} - ab\vec{v} = 0.$$

On the other hand,

$$b(V_2 - V_1) + a(V_3 - V_1) = (-a-b)V_1 + bV_2 + aV_3.$$

So $(-a-b)\vec{V}_1 + b\vec{V}_2 + a\vec{V}_3 = \vec{0}$, $(**)$ and the coeff a of $\vec{V}_3$ is not zero. The equations $(*)$ and $(**)$ show that the set $\{V_1, V_2, V_3\}$ is linearly dependent.

Note: The original formulation of this problem used row vectors $\vec{x}, \vec{u}$ and the dot product was $\vec{x}\,\vec{u}^T$. The two formulations are equivalent.

6. Let $\vec{u}$ be a non-zero vector in $\mathbb{R}^2$. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be given by the formula $R(\vec{x}) = 2\left(\frac{\vec{x}^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u} - \vec{x}$. Note that vectors in $\mathbb{R}^2$ are considered here as 2×1 matrices, and we regard the 1×1 matrices $\vec{x}^T\vec{u}$ and $\vec{u}^T\vec{u}$ as scalars, so that the fraction $\frac{\vec{x}^T\vec{u}}{\vec{u}^T\vec{u}}$ is a scalar (quotient of two dot products).

(a) Show that R is a linear transformation by verifying properties (1) and (2) in the definition of a linear transformation in Section 1.8.

(1)
$$R(\vec{x}+\vec{y}) = 2\left(\frac{(\vec{x}+\vec{y})^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u} - (\vec{x}+\vec{y}) \underset{\uparrow}{=} 2\left(\frac{\vec{x}^T\vec{u}}{\vec{u}^T\vec{u}} + \frac{\vec{y}^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u} - \vec{x} - \vec{y} = R(\vec{x}) + R(\vec{y}).$$

$\boxed{(\vec{x}+\vec{y})^T = \vec{x}^T + \vec{y}^T \text{ and distributive property of matrix multiplication}}$

(2)
$$R(c\vec{x}) = 2\left(\frac{(c\vec{x})^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u} - c\vec{x} = c\left[2\left(\frac{\vec{x}^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u} - \vec{x}\right] = cR(\vec{x}).$$

(b) Let L be the line span$\{\vec{u}\}$ in $\mathbb{R}^2$. Show that $R(\vec{v}) = \vec{v}$, for every vector $\vec{v}$ in L.

If $\vec{v}$ is in L, then $\vec{v} = c\vec{u}$ for some scalar c. So
$$R(\vec{v}) = R(c\vec{u}) = cR(\vec{u}) = c\left[2\Big(\underbrace{\frac{\vec{u}^T\vec{u}}{\vec{u}^T\vec{u}}}_{1}\Big)\vec{u} - \vec{u}\right] = c\vec{u} = \vec{v}.$$

(c) Two vectors $\vec{v}$, $\vec{w}$ in $\mathbb{R}^2$ are orthogonal (i.e., perpendicular), if $\vec{v}^T\vec{w} = 0$. Show that $R(\vec{x}) = -\vec{x}$, if $\vec{x}$ is orthogonal to $\vec{u}$. *Note: Parts 6b and 6c show that R is the reflection of $\mathbb{R}^2$ with respect to the line L.*

Assume $\vec{x}^T\vec{u} = \vec{0}$. Then
$$R(\vec{x}) = 2\left(\frac{\vec{x}^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u} - \vec{x} = 2 \cdot 0 \cdot \vec{u} - \vec{x} = -\vec{x}.$$

(d) Let $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $L = \text{span}\{\vec{u}\}$ as above. Find the standard matrix A of the reflection $R : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to L. $A = \left(R\begin{pmatrix}1\\0\end{pmatrix}\ R\begin{pmatrix}0\\1\end{pmatrix}\right)$

$$R\begin{pmatrix}1\\0\end{pmatrix} = 2\left(\frac{(1,0)\begin{pmatrix}2\\3\end{pmatrix}}{(2\ 3)\begin{pmatrix}2\\3\end{pmatrix}}\right)\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \frac{4}{13}\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-5/13\\12/13\end{pmatrix}$$

$$R\begin{pmatrix}0\\1\end{pmatrix} = 2\left(\frac{(0\ 1)\begin{pmatrix}2\\3\end{pmatrix}}{(2\ 3)\begin{pmatrix}2\\3\end{pmatrix}}\right)\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \frac{6}{13}\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}12/13\\5/13\end{pmatrix}.$$

So $A = \left(\frac{1}{13}\right)\begin{pmatrix}-5 & 12\\12 & 5\end{pmatrix}.$

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