Handwritten HW 1

1) False. There does NOT exist a 2×2 matrix A such $A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$ and $A\begin{pmatrix}3\\3\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$ because

$$A\begin{pmatrix}3\\3\end{pmatrix}=A\left(3\begin{pmatrix}1\\1\end{pmatrix}\right)\underset{\uparrow}{=}3A\begin{pmatrix}1\\1\end{pmatrix}=3\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}9\\3\end{pmatrix}\neq\begin{pmatrix}1\\3\end{pmatrix},$$

2-nd linearity property of multiplication of a vector by a matrix

2) False. There do NOT exist scalars a, b such that the column of the matrix

$$A=\begin{pmatrix}0&1&a\\-1&0&b\\-a&-b&0\end{pmatrix}$$

span the whole of $\mathbb{R}^3$,

because the echelon forms of A do not have a pivot in every row. Indeed

$$A\underset{\substack{\uparrow\\ \text{interchange}\\ R1 \text{ and } R2}}{\sim}\begin{pmatrix}-1&0&b\\0&1&a\\-a&-b&0\end{pmatrix}\underset{\substack{\bullet\text{Multiply } R1\\ \text{by } -1\\ \bullet\text{Add } aR1 \text{ to } R_3}}{\sim}\begin{pmatrix}1&0&-b\\0&1&a\\0&-b&ab\end{pmatrix}\overset{\text{Add } bR2 \text{ to } R_3}{\sim}$$

$$\begin{pmatrix}1&0&-b\\0&1&a\\0&0&0\end{pmatrix}$$

only two pivot positions,

3) True!

If $\vec{u} = c_1 \vec{v} + c_2 \vec{w}$ and

$\vec{v} = a_1 \vec{p} + a_2 \vec{q} + a_3 \vec{r}$, then

$$\vec{u} = c_1(a_1\vec{p} + a_2\vec{q} + a_3\vec{r}) + c_2\vec{w} =$$

$$= (c_1 a_1)\vec{p} + (c_1 a_2)\vec{q} + (c_1 a_3)\vec{r} + c_2\vec{w}$$

So $\vec{u}$ is a linear combination of $\vec{p}, \vec{q}, \vec{r}$ and $\vec{w}$.

4) True. If A is a 4×3 matrix, then it has at most 3 pivot positions, since each column contains at most one pivot position. Hence, the fourth row of an echelon form of A must be a zero row. So A does not have a pivot position in every row. So there exists a vector $\vec{b}$ in $\mathbb{R}^4$ such that $A\vec{x} = \vec{b}$ is inconsistent.

True.

5) If A is a 4×3 matrix with a pivot position in every column, then the homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

If $A\vec{v} = A\vec{w}$, then

$$A(\vec{v} - \vec{w}) = A\vec{v} - A\vec{w} = \vec{0},$$

So $\vec{v} - \vec{w} = \vec{0}$ and hence $\vec{v} = \vec{w}$.

6) True. $A\vec{x} = \vec{0}$ has only the trivial sol'n implies that A has a pivot position in every column. Since A is an $n \times n$ square matrix, the row reduced echelon form of A is the $n \times n$ identity matrix I_n. Similarly the row reduced echelon form of B is I_n. Now, elementary row operations are reversible, so there is a finite sequence of elementary row operation taking I_n (back) to B. Performing these after the

$$A \underset{\text{E.R.O}}{\overset{\text{finite seq of}}{\sim\!\!\to}} I_n \underset{\text{seq of E.R.O}}{\overset{\text{finite}}{\sim\!\!\to}} B$$

row reduction of A to I_n we get a sequence of E.R.O's taking A to B.

7) True.

If $B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ is the row reduced

echelon form of $A = (\vec{u}\ \vec{v}\ \vec{w})$, then

$A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the

same set of solutions. Now

$B\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = A\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 3\vec{u} + 2\vec{v} - \vec{w}$.

So $\vec{w} = 3u + 2\vec{v}$.

8) True.

If $\vec{w}$ belongs to span$\{\vec{t}, \vec{u}, \vec{v}\}$ then there exist scalars c_1, c_2, c_3, such that

(*) $\vec{w} = c_1\vec{t} + c_2\vec{u} + c_3\vec{v}$.

If $c_3 = 0$, then $\vec{w}$ belongs to Span$\{\vec{t}, \vec{u}\}$.

We are given that $\vec{w}$ does not " " ".

Hence, $c_3 \neq 0$. Thus, we can divide both sides of (*) by c_3 and solve for $\vec{v}$ to get $\vec{v} = \left(\frac{1}{c_3}\right)\vec{w} - \left(\frac{c_1}{c_3}\right)\vec{t} - \left(\frac{c_2}{c_3}\right)\vec{u}$.

So $\vec{v}$ belongs to span$\{\vec{t}, \vec{u}, \vec{w}\}$.